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On the Continuity of Some Functions

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. We prove that basic arithmetic operations preserve continuity of functions.

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The terminology and notation used here have been introduced in the following articles: [20], [1], [6], [13], [4], [7], [19], [8], [9], [5], [21], [2], [3], [10], [18], [25], [26], [23], [12], [22], [24], [14], [16], [17], [15], and [11].

1. Preliminaries

For simplicity, we adopt the following rules: x, X are sets, i, n, m are natural numbers, r, s are real numbers, c, c_1, c_2, d are complex numbers, f, g are complex-valued functions, g_1 is an n-element complex-valued finite sequence, f_1 is an n-element real-valued finite sequence, T is a non empty topological space, and p is an element of $\mathcal{E}^n_{\mathrm{T}}$.

Let R be a binary relation and let X be an empty set. Observe that $R^\circ X$ is empty and $R^{-1}(X)$ is empty.

Let A be an empty set. Observe that every element of A is empty.

We now state the proposition

(1) For every trivial set X and for every set Y such that $X \approx Y$ holds Y is trivial.

Let r be a real number. Observe that $r^{\mathbf{2}}$ is non negative.

Let r be a positive real number. Note that $r^{\mathbf{2}}$ is positive.

Let us note that $\sqrt{0}$ is zero.

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Let f be an empty set. Note that ${}^2f$ is empty and $|f|$ is zero.

The following propositions are true:

(2) $f\,(c_1 + c_2) = f\,c_1 + f\,c_2$.

(3) $f\,(c_1 - c_2) = f\,c_1 - f\,c_2$.

(4) $f/c + g/c = (f + g)/c$.

(5) $f/c - g/c = (f - g)/c$.

(6) If $c_1 \neq 0$ and $c_2 \neq 0$, then $f/c_1 - g/c_2 = (f\,c_2 - g\,c_1)/(c_1 \cdot c_2)$.

(7) If $c \neq 0$, then $f/c - g = (f - c\,g)/c$.

(8) $(c - d)\,f = c\,f - d\,f$.

(9) $(f - g)^{\mathbf{2}} = (g - f)^{\mathbf{2}}$.

(10) $(f/c)^{\mathbf{2}} = f^{\mathbf{2}}/c^{\mathbf{2}}$.

(11) $|n \mapsto r - n \mapsto s| = \sqrt{n} \cdot |r - s|$.

Let us consider f, x, c. Observe that $f +\cdot\, (x, c)$ is complex-valued.

We now state a number of propositions:

(12) $(\langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, c))^{\mathbf{2}} = \langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, c^{\mathbf{2}})$.

(13) If $x \in \operatorname{Seg} n$, then $|\langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (x, r)| = |r|$.

(14) $0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, 0) = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(15) $f_1 \bullet (0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, r)) = 0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, f_1(x) \cdot r)$.

(16) $|(f_1, 0_{\mathcal{E}^n_{\mathrm{T}}} +\cdot\, (x, r))| = f_1(x) \cdot r$.

(17) $(g_1 +\cdot\, (i, c)) - g_1 = \langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot\, (i, c - g_1(i))$.

(18) $|\langle r \rangle| = |r|$.

(19) Every real-valued finite sequence is a finite sequence of elements of $\mathbb{R}$.

(20) For every real-valued finite sequence f such that $|f| \neq 0$ there exists a natural number i such that $i \in \operatorname{dom} f$ and $f(i) \neq 0$.

(21) For every real-valued finite sequence f holds $|\sum f| \leq \sum |f|$.

(22) Let A be a non empty 1-sorted structure, B be a trivial non empty 1-sorted structure, t be a point of B, and f be a function from A into B. Then $f = A \longmapsto t$.

Let n be a non zero natural number, let i be an element of $\operatorname{Seg} n$, and let T be a real-membered non empty topological space. Note that $\operatorname{proj}(\operatorname{Seg} n \longmapsto T, i)$ is real-valued.

Let us consider n, let p be an element of $\mathcal{R}^n$, and let us consider r. Then p/r is an element of $\mathcal{R}^n$.

One can prove the following proposition

(23) For all points p, q of $\mathcal{E}^m_{\mathrm{T}}$ holds $p \in \operatorname{Ball}(q, r)$ iff $-p \in \operatorname{Ball}(-q, r)$.

Let S be a 1-sorted structure. We say that S is complex-functions-membered if and only if:

(Def. 1) The carrier of S is complex-functions-membered.

We say that S is real-functions-membered if and only if:

(Def. 2) The carrier of S is real-functions-membered.

Let us consider n. One can verify that $\mathcal{E}^n_{\mathrm{T}}$ is real-functions-membered.

Let us observe that $\mathcal{E}^0_{\mathrm{T}}$ is real-membered.

One can check that $\mathcal{E}^0_{\mathrm{T}}$ is trivial.

Let us observe that every 1-sorted structure which is real-functions-membered is also complex-functions-membered.

Let us mention that there exists a 1-sorted structure which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered 1-sorted structure. One can check that the carrier of S is complex-functions-membered.

Let S be a real-functions-membered 1-sorted structure. Note that the carrier of S is real-functions-membered.

Let us observe that there exists a topological space which is strict, non empty, and real-functions-membered.

Let S be a complex-functions-membered topological space. Observe that every subspace of S is complex-functions-membered.

Let S be a real-functions-membered topological space. One can verify that every subspace of S is real-functions-membered.

Let X be a complex-functions-membered set. The functor $(-)X$ yields a complex-functions-membered set and is defined as follows:

(Def. 3) For every complex-valued function f holds $-f \in (-)X$ iff $f \in X$.

Let us observe that the functor $(-)X$ is involutive.

Let X be an empty set. One can verify that $(-)X$ is empty.

Let X be a non empty complex-functions-membered set. Observe that $(-)X$ is non empty.

The following proposition is true

(24) Let X be a complex-functions-membered set and f be a complex-valued function. Then $-f \in X$ if and only if $f \in (-)X$.

Let X be a real-functions-membered set. One can verify that $(-)X$ is real-functions-membered.

Next we state the proposition

(25) For every subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $-X = (-)X$.

Let us consider n and let X be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $(-)X$ is a subset of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n and let X be an open subset of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $(-)X$ is open.

Let us consider n, p, x. Then $p(x)$ is an element of $\mathbb{R}$.

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T, and let x be a point of $R \times S$. Then $f(x)$ is a point of T.

Let R, S, T be non empty topological spaces, let f be a function from $R \times S$ into T, let r be a point of R, and let s be a point of S. Then $f(r, s)$ is a point of T.

Let us consider n, p, r. Then $p + r$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n, p, r. Then $p - r$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n, p, r. Then $p\,r$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n, p, r. Then p/r is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then $p_1\,p_2$ is a point of $\mathcal{E}^n_{\mathrm{T}}$. Let us note that the functor $p_1\,p_2$ is commutative.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Then ${}^2p$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then p_1/p_2 is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n, p, x, r. Then $p +\cdot (x, r)$ is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Next we state the proposition

(26) For all points a, o of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \neq 0$ and $a \in \mathrm{Ball}(o, r)$ holds $|\sum(a - o)| < n \cdot r$.

Let us consider n. Note that $\mathcal{E}^n$ is real-functions-membered.

One can prove the following propositions:

(27) Let V be an add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V. Then $(v+u)-u = v$.

(28) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure and v, u be elements of V. Then $(v - u) + u = v$.

(29) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f + c = f + (\mathrm{dom}\, f \longmapsto c)$.

(30) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f - c = f - (\mathrm{dom}\, f \longmapsto c)$.

(31) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f \cdot c = f \cdot (\mathrm{dom}\, f \longmapsto c)$.

(32) For every complex-functions-membered set Y and for every partial function f from X to Y holds $f/c = f/(\mathrm{dom}\, f \longmapsto c)$.

Let D be a complex-functions-membered set and let f, g be finite sequences of elements of D. One can verify the following observations:

* $f + g$ is finite sequence-like,
* $f - g$ is finite sequence-like,
* $f \cdot g$ is finite sequence-like, and
* f/g is finite sequence-like.

Next we state a number of propositions:

(33) For every function f from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $-f$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(34) For every function f from $\mathcal{E}^i_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ holds $f \circ -$ is a function from $\mathcal{E}^i_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$.

(35) For every function f from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f + r$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(36) For every function f from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f - r$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(37) For every function f from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f \cdot r$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(38) For every function f from X into $\mathcal{E}^n_{\mathrm{T}}$ holds f/r is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(39) For all functions f, g from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f + g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(40) For all functions f, g from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f - g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(41) For all functions f, g from X into $\mathcal{E}^n_{\mathrm{T}}$ holds $f \cdot g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(42) For all functions f, g from X into $\mathcal{E}^n_{\mathrm{T}}$ holds f/g is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(43) Let f be a function from X into $\mathcal{E}^n_{\mathrm{T}}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f + g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(44) Let f be a function from X into $\mathcal{E}^n_{\mathrm{T}}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f - g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(45) Let f be a function from X into $\mathcal{E}^n_{\mathrm{T}}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then $f \cdot g$ is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

(46) Let f be a function from X into $\mathcal{E}^n_{\mathrm{T}}$ and g be a function from X into $\mathbb{R}^{\mathbf{1}}$. Then f/g is a function from X into $\mathcal{E}^n_{\mathrm{T}}$.

Let n be a natural number, let T be a non empty set, let R be a real-membered set, and let f be a function from T into R. The functor $\mathrm{incl}(f, n)$ yields a function from T into $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 4) For every element t of T holds $(\mathrm{incl}(f, n))(t) = n \mapsto f(t)$.

We now state several propositions:

(47) Let R be a real-membered set, f be a function from T into R, and t be a point of T. If $x \in \mathrm{Seg}\, n$, then $(\mathrm{incl}(f, n))(t)(x) = f(t)$.

(48) For every non empty set T and for every real-membered set R and for every function f from T into R holds $\mathrm{incl}(f, 0) = T \longmapsto 0$.

(49) For every function f from T into $\mathcal{E}^n_{\mathrm{T}}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f + g = f + \mathrm{incl}(g, n)$.

(50) For every function f from T into $\mathcal{E}^n_{\mathrm{T}}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f - g = f - \mathrm{incl}(g, n)$.

(51) For every function f from T into $\mathcal{E}^n_{\mathrm{T}}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f \cdot g = f \cdot \mathrm{incl}(g, n)$.

(52) For every function f from T into $\mathcal{E}^n_{\mathrm{T}}$ and for every function g from T into $\mathbb{R}^{\mathbf{1}}$ holds $f/g = f/\,\mathrm{incl}(g, n)$.

Let us consider n. The functor $\bigotimes_n$ yields a function from $\mathcal{E}^n_{\mathrm{T}} \times \mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 5) For all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ holds $\bigotimes_n(x, y) = x\,y$.

Next we state two propositions:

(53) $\bigotimes_0 = \mathcal{E}^0_{\mathrm{T}} \times \mathcal{E}^0_{\mathrm{T}} \longmapsto 0_{\mathcal{E}^0_{\mathrm{T}}}$.

(54) For all functions f, g from T into $\mathcal{E}^n_{\mathrm{T}}$ holds $f \cdot g = (\bigotimes_n)^\circ(f, g)$.

Let us consider m, n. The functor $\mathrm{PROJ}(m, n)$ yields a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined as follows:

(Def. 6) For every element p of $\mathcal{E}^m_{\mathrm{T}}$ holds $(\mathrm{PROJ}(m, n))(p) = p_n$.

One can prove the following propositions:

(55) For every point p of $\mathcal{E}^m_{\mathrm{T}}$ such that $n \in \mathrm{dom}\, p$ holds $(\mathrm{PROJ}(m, n))^\circ\,\mathrm{Ball}(p, r) =]p_n - r, p_n + r[$.

(56) For every non zero natural number m and for every function f from T into $\mathbb{R}^{\mathbf{1}}$ holds $f = \mathrm{PROJ}(m, m) \cdot \mathrm{incl}(f, m)$.

2. Continuity

Let us consider T. One can check that there exists a function from T into $\mathbb{R}^{\mathbf{1}}$ which is non-empty and continuous.

Next we state two propositions:

(57) If $n \in \mathrm{Seg}\, m$, then $\mathrm{PROJ}(m, n)$ is continuous.

(58) If $n \in \mathrm{Seg}\, m$, then $\mathrm{PROJ}(m, n)$ is open.

Let us consider n, T and let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that $\mathrm{incl}(f, n)$ is continuous.

Let us consider n. One can verify that $\bigotimes_n$ is continuous.

One can prove the following proposition

(59) Let f be a function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is continuous. Then $f \circ -$ is a continuous function from $\mathcal{E}^m_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider T and let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that $-f$ is continuous.

Let us consider T and let f be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. One can verify that f^{-1} is continuous.

Let us consider T, let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$, and let us consider r. One can check the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous,
* $f\,r$ is continuous, and
* f/r is continuous.

Let us consider T and let f, g be continuous functions from T into $\mathbb{R}^{\mathbf{1}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f\,g$ is continuous.

Let us consider T, let f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$, and let g be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that f/g is continuous.

Let us consider n, T and let f, g be continuous functions from T into $\mathcal{E}^n_{\mathrm{T}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider n, T, let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$, and let g be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. One can verify the following observations:

* $f + g$ is continuous,
* $f - g$ is continuous, and
* $f \cdot g$ is continuous.

Let us consider n, T, let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$, and let g be a non-empty continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Observe that f/g is continuous.

Let us consider n, T, r and let f be a continuous function from T into $\mathcal{E}^n_{\mathrm{T}}$. One can verify the following observations:

* $f + r$ is continuous,
* $f - r$ is continuous,
* $f \cdot r$ is continuous, and
* f/r is continuous.

We now state two propositions:

(60) Let r be a non negative real number, n be a non zero natural number, and p be a point of $\mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$. Then $-p$ is a point of $\mathrm{Tcircle}(0_{\mathcal{E}^n_{\mathrm{T}}}, r)$.

(61) Let r be a non negative real number and f be a function from $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{n}$. Then $f \circ -$ is a function from $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{n}$.

Let n be a natural number, let r be a non negative real number, and let X be a subset of $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$. Then $(-)X$ is a subset of $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{n+1}}, r)$.

Let us consider m, let r be a non negative real number, and let X be an open subset of $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$. One can verify that $(-)X$ is open.

The following proposition is true

(62) Let r be a non negative real number and f be a continuous function from $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{m}$. Then $f \circ -$ is a continuous function from $\mathrm{Tcircle}(0_{\mathcal{E}_{\mathrm{T}}^{m+1}}, r)$ into $\mathcal{E}_{\mathrm{T}}^{m}$.

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