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Affine Independence in Vector Spaces

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Summary. In this article we describe the notion of affinely independent subset of a real linear space. First we prove selected theorems concerning operations on linear combinations. Then we introduce affine independence and prove the equivalence of various definitions of this notion. We also introduce the notion of the affine hull, i.e. a subset generated by a set of vectors which is an intersection of all affine sets including the given set. Finally, we introduce and prove selected properties of the barycentric coordinates.

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The terminology and notation used here are introduced in the following papers: [1], [6], [10], [2], [3], [8], [15], [13], [12], [11], [7], [5], [9], [14], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: x, y are sets, r, s are real numbers, S is a non empty additive loop structure, L_1, L_2, L_3 are linear combinations of S, G is an Abelian add-associative right zeroed right complementable non empty additive loop structure, L_4, L_5, L_6 are linear combinations of G, g, h are elements of G, R_1 is a non empty RLS structure, R is a real linear space-like non empty RLS structure, A_1 is a subset of R, L_7, L_8, L_9 are linear combinations of R, V is a real linear space, v, v_1, v_2, w, p are vectors of V, A, B are subsets of V, F_1, F_2 are families of subsets of V, and L, L_{10}, L_{11} are linear combinations of V.

Let us consider R_1 and let A be an empty subset of R_1. Note that conv A is empty.

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Let us consider R_1 and let A be a non empty subset of R_1. One can check that $\operatorname{conv} A$ is non empty.

One can prove the following propositions:

(1) For every element v of R holds $\operatorname{conv}\{v\} = \{v\}$.

(2) For every subset A of R_1 holds $A \subseteq \operatorname{conv} A$.

(3) For all subsets A, B of R_1 such that $A \subseteq B$ holds $\operatorname{conv} A \subseteq \operatorname{conv} B$.

(4) For all subsets S, A of R_1 such that $A \subseteq \operatorname{conv} S$ holds $\operatorname{conv} S = \operatorname{conv} S \cup A$.

(5) Let V be an add-associative non empty additive loop structure, A be a subset of V, and v, w be elements of V. Then $(v + w) + A = v + (w + A)$.

(6) For every Abelian right zeroed non empty additive loop structure V and for every subset A of V holds $0_V + A = A$.

(7) For every subset A of G holds $\operatorname{Card} A = \operatorname{Card}(g + A)$.

(8) For every element v of S holds $v + \emptyset_S = \emptyset_S$.

(9) For all subsets A, B of R_1 such that $A \subseteq B$ holds $r \cdot A \subseteq r \cdot B$.

(10) $(r \cdot s) \cdot A_1 = r \cdot (s \cdot A_1)$.

(11) $1 \cdot A_1 = A_1$.

(12) $0 \cdot A \subseteq \{0_V\}$.

(13) For every finite sequence F of elements of S holds $(L_2 + L_3) \cdot F = L_2 \cdot F + L_3 \cdot F$.

(14) For every finite sequence F of elements of V holds $(r \cdot L) \cdot F = r \cdot (L \cdot F)$.

(15) Suppose A is linearly independent and $A \subseteq B$ and $\operatorname{Lin}(B) = V$. Then there exists a linearly independent subset I of V such that $A \subseteq I \subseteq B$ and $\operatorname{Lin}(I) = V$.

2. Two Transformations of Linear Combinations

Let us consider G, L_4, g. The functor $g + L_4$ yielding a linear combination of G is defined as follows:

(Def. 1) $(g + L_4)(h) = L_4(h - g)$.

Next we state several propositions:

(16) The support of $g + L_4 = g +$ the support of L_4.

(17) $g + (L_5 + L_6) = (g + L_5) + (g + L_6)$.

(18) $v + r \cdot L = r \cdot (v + L)$.

(19) $g + (h + L_4) = (g + h) + L_4$.

(20) $g + \mathbf{0}_{\mathrm{LC}_G} = \mathbf{0}_{\mathrm{LC}_G}$.

(21) $0_G + L_4 = L_4$.

Let us consider R, L_7, r. The functor $r \circ L_7$ yields a linear combination of R and is defined as follows:

(Def. 2)(i) For every element v of R holds $(r \circ L_7)(v) = L_7(r^{-1} \cdot v)$ if $r \neq 0$,

(ii) $r \circ L_7 = \mathbf{0}_{\mathrm{LC}_R}$, otherwise.

The following propositions are true:

(22) The support of $r \circ L_7 \subseteq r \cdot$ (the support of L_7).

(23) If $r \neq 0$, then the support of $r \circ L_7 = r \cdot$ (the support of L_7).

(24) $r \circ (L_8 + L_9) = r \circ L_8 + r \circ L_9$.

(25) $r \cdot (s \circ L) = s \circ (r \cdot L)$.

(26) $r \circ \mathbf{0}_{\mathrm{LC}_R} = \mathbf{0}_{\mathrm{LC}_R}$.

(27) $r \circ (s \circ L_7) = (r \cdot s) \circ L_7$.

(28) $1 \circ L_7 = L_7$.

3. The Sum of Coefficients of a Linear Combination

Let us consider S, L_1. The functor $\mathrm{sum}\, L_1$ yields a real number and is defined as follows:

(Def. 3) There exists a finite sequence F of elements of S such that F is one-to-one and $\mathrm{rng}\, F =$ the support of L_1 and $\mathrm{sum}\, L_1 = \sum(L_1 \cdot F)$.

One can prove the following propositions:

(29) For every finite sequence F of elements of S such that the support of L_1 misses $\mathrm{rng}\, F$ holds $\sum(L_1 \cdot F) = 0$.

(30) Let F be a finite sequence of elements of S. If F is one-to-one and the support of $L_1 \subseteq \mathrm{rng}\, F$, then $\mathrm{sum}\, L_1 = \sum(L_1 \cdot F)$.

(31) $\mathrm{sum}\, \mathbf{0}_{\mathrm{LC}_S} = 0$.

(32) For every element v of S such that the support of $L_1 \subseteq \{v\}$ holds $\mathrm{sum}\, L_1 = L_1(v)$.

(33) For all elements v_1, v_2 of S such that the support of $L_1 \subseteq \{v_1, v_2\}$ and $v_1 \neq v_2$ holds $\mathrm{sum}\, L_1 = L_1(v_1) + L_1(v_2)$.

(34) $\mathrm{sum}\, L_2 + L_3 = \mathrm{sum}\, L_2 + \mathrm{sum}\, L_3$.

(35) $\mathrm{sum}\, r \cdot L = r \cdot \mathrm{sum}\, L$.

(36) $\mathrm{sum}\, L_{10} - L_{11} = \mathrm{sum}\, L_{10} - \mathrm{sum}\, L_{11}$.

(37) $\mathrm{sum}\, L_4 = \mathrm{sum}\, g + L_4$.

(38) If $r \neq 0$, then $\mathrm{sum}\, L_7 = \mathrm{sum}\, r \circ L_7$.

(39) $\sum(v + L) = \mathrm{sum}\, L \cdot v + \sum L$.

(40) $\sum(r \circ L) = r \cdot \sum L$.

4. Affine Independence of Vectors

Let us consider V, A. We say that A is affinely independent if and only if:

(Def. 4) A is empty or there exists v such that $v \in A$ and $(-v + A) \setminus \{0_V\}$ is linearly independent.

Let us consider V. Observe that every subset of V which is empty is also affinely independent. Let us consider v. One can check that $\{v\}$ is affinely independent. Let us consider w. Observe that $\{v, w\}$ is affinely independent.

Let us consider V. Note that there exists a subset of V which is non empty, trivial, and affinely independent.

We now state three propositions:

(41) A is affinely independent iff for every v such that $v \in A$ holds $(-v+A) \setminus \{0_V\}$ is linearly independent.

(42) A is affinely independent if and only if for every linear combination L of A such that $\sum L = 0_V$ and $\operatorname{sum} L = 0$ holds the support of $L = \emptyset$.

(43) If A is affinely independent and $B \subseteq A$, then B is affinely independent.

Let us consider V. Note that every subset of V which is linearly independent is also affinely independent.

In the sequel I denotes an affinely independent subset of V.

Let us consider V, I, v. Observe that $v + I$ is affinely independent.

One can prove the following proposition

(44) If $v + A$ is affinely independent, then A is affinely independent.

Let us consider V, I, r. One can check that $r \cdot I$ is affinely independent.

The following propositions are true:

(45) If $r \cdot A$ is affinely independent and $r \neq 0$, then A is affinely independent.

(46) If $0_V \in A$, then A is affinely independent iff $A \setminus \{0_V\}$ is linearly independent.

Let us consider V and let F be a family of subsets of V. We say that F is affinely independent if and only if:

(Def. 5) If $A \in F$, then A is affinely independent.

Let us consider V. Observe that every family of subsets of V which is empty is also affinely independent. Let us consider I. One can check that $\{I\}$ is affinely independent.

Let us consider V. Note that there exists a family of subsets of V which is empty and affinely independent and there exists a family of subsets of V which is non empty and affinely independent.

Next we state two propositions:

(47) If F_1 is affinely independent and F_2 is affinely independent, then $F_1 \cup F_2$ is affinely independent.

(48) If $F_1 \subseteq F_2$ and F_2 is affinely independent, then F_1 is affinely independent.

5. Affine Hull

Let us consider R_1 and let A be a subset of R_1. The functor $\mathrm{Affin}\, A$ yields a subset of R_1 and is defined as follows:

(Def. 6) $\mathrm{Affin}\, A = \bigcap\{B; B$ ranges over affine subsets of R_1: $A \subseteq B\}$.

Let us consider R_1 and let A be a subset of R_1. Observe that $\mathrm{Affin}\, A$ is affine.

Let us consider R_1 and let A be an empty subset of R_1. Note that $\mathrm{Affin}\, A$ is empty.

Let us consider R_1 and let A be a non empty subset of R_1. Note that $\mathrm{Affin}\, A$ is non empty.

One can prove the following propositions:

(49) For every subset A of R_1 holds $A \subseteq \mathrm{Affin}\, A$.

(50) For every affine subset A of R_1 holds $A = \mathrm{Affin}\, A$.

(51) For all subsets A, B of R_1 such that $A \subseteq B$ and B is affine holds $\mathrm{Affin}\, A \subseteq B$.

(52) For all subsets A, B of R_1 such that $A \subseteq B$ holds $\mathrm{Affin}\, A \subseteq \mathrm{Affin}\, B$.

(53) $\mathrm{Affin}(v + A) = v + \mathrm{Affin}\, A$.

(54) If A_1 is affine, then $r \cdot A_1$ is affine.

(55) If $r \neq 0$, then $\mathrm{Affin}(r \cdot A_1) = r \cdot \mathrm{Affin}\, A_1$.

(56) $\mathrm{Affin}(r \cdot A) = r \cdot \mathrm{Affin}\, A$.

(57) If $v \in \mathrm{Affin}\, A$, then $\mathrm{Affin}\, A = v + \mathrm{Up}(\mathrm{Lin}(-v + A))$.

(58) A is affinely independent iff for every B such that $B \subseteq A$ and $\mathrm{Affin}\, A = \mathrm{Affin}\, B$ holds $A = B$.

(59) $\mathrm{Affin}\, A = \{\sum L; L$ ranges over linear combinations of A: $\mathrm{sum}\, L = 1\}$.

(60) If $I \subseteq A$, then there exists an affinely independent subset I_1 of V such that $I \subseteq I_1 \subseteq A$ and $\mathrm{Affin}\, I_1 = \mathrm{Affin}\, A$.

(61) Let A, B be finite subsets of V. Suppose A is affinely independent and $\mathrm{Affin}\, A = \mathrm{Affin}\, B$ and $\overline{\overline{B}} \leq \overline{\overline{A}}$. Then B is affinely independent.

(62) L is convex iff $\mathrm{sum}\, L = 1$ and for every v holds $0 \leq L(v)$.

(63) If L is convex, then $L(x) \leq 1$.

(64) If L is convex and $L(x) = 1$, then the support of $L = \{x\}$.

(65) $\mathrm{conv}\, A \subseteq \mathrm{Affin}\, A$.

(66) If $x \in \mathrm{conv}\, A$ and $\mathrm{conv}\, A \setminus \{x\}$ is convex, then $x \in A$.

(67) $\mathrm{Affin}\, \mathrm{conv}\, A = \mathrm{Affin}\, A$.

(68) If $\mathrm{conv}\, A \subseteq \mathrm{conv}\, B$, then $\mathrm{Affin}\, A \subseteq \mathrm{Affin}\, B$.

(69) For all subsets A, B of R_1 such that $A \subseteq \mathrm{Affin}\, B$ holds $\mathrm{Affin}(A \cup B) = \mathrm{Affin}\, B$.

6. Barycentric Coordinates

Let us consider V and let us consider A. Let us assume that A is affinely independent. Let us consider x. Let us assume that $x \in \operatorname{Affin} A$. The functor $x \to A$ yielding a linear combination of A is defined by:

(Def. 7) $\sum(x \to A) = x$ and $\operatorname{sum} x \to A = 1$.

We now state a number of propositions:

(70) If $v_1, v_2 \in \operatorname{Affin} I$, then $(1-r) \cdot v_1 + r \cdot v_2 \to I = (1-r) \cdot (v_1 \to I) + r \cdot (v_2 \to I)$.

(71) If $x \in \operatorname{conv} I$, then $x \to I$ is convex and $0 \leq (x \to I)(v) \leq 1$.

(72) If $x \in \operatorname{conv} I$, then $(x \to I)(y) = 1$ iff $x = y$ and $x \in I$.

(73) For every I such that $x \in \operatorname{Affin} I$ and for every v such that $v \in I$ holds $0 \leq (x \to I)(v)$ holds $x \in \operatorname{conv} I$.

(74) If $x \in I$, then $\operatorname{conv} I \setminus \{x\}$ is convex.

(75) For every B such that $x \in \operatorname{Affin} I$ and for every y such that $y \in B$ holds $(x \to I)(y) = 0$ holds $x \in \operatorname{Affin}(I \setminus B)$ and $x \to I = x \to I \setminus B$.

(76) For every B such that $x \in \operatorname{conv} I$ and for every y such that $y \in B$ holds $(x \to I)(y) = 0$ holds $x \in \operatorname{conv} I \setminus B$.

(77) If $B \subseteq I$ and $x \in \operatorname{Affin} B$, then $x \to B = x \to I$.

(78) If $v_1, v_2 \in \operatorname{Affin} A$ and $r + s = 1$, then $r \cdot v_1 + s \cdot v_2 \in \operatorname{Affin} A$.

(79) For all finite subsets A, B of V such that A is affinely independent and $\operatorname{Affin} A \subseteq \operatorname{Affin} B$ holds $\overline{\overline{A}} \leq \overline{\overline{B}}$.

(80) Let A, B be finite subsets of V. Suppose A is affinely independent and $\operatorname{Affin} A \subseteq \operatorname{Affin} B$ and $\overline{\overline{A}} = \overline{\overline{B}}$. Then B is affinely independent.

(81) If $L_{10}(v) \neq L_{11}(v)$, then $(r \cdot L_{10} + (1-r) \cdot L_{11})(v) = s$ iff $r = \frac{L_{11}(v)-s}{L_{11}(v)-L_{10}(v)}$.

(82) $A \cup \{v\}$ is affinely independent iff A is affinely independent but $v \in A$ or $v \notin \operatorname{Affin} A$.

(83) If $w \notin \operatorname{Affin} A$ and $v_1, v_2 \in A$ and $r \neq 1$ and $r \cdot w + (1-r) \cdot v_1 = s \cdot w + (1-s) \cdot v_2$, then $r = s$ and $v_1 = v_2$.

(84) If $v \in I$ and $w \in \operatorname{Affin} I$ and $p \in \operatorname{Affin}(I \setminus \{v\})$ and $w = r \cdot v + (1-r) \cdot p$, then $r = (w \to I)(v)$.

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