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Fixpoint Theorem for Continuous Functions on Chain-Complete Posets

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Summary. This text includes the definition of chain-complete poset, fixpoint theorem on it, and the definition of the function space of continuous functions on chain-complete posets [10].

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The papers [8], [4], [5], [3], [1], [9], [7], [11], [13], [12], [2], [14], and [6] provide the notation and terminology for this paper.

1. Monotone Functions, Chain and Chain-complete Posets

Let P be a non empty poset. Observe that there exists a chain of P which is non empty.

Let I_1 be a relational structure. We say that I_1 is chain-complete if and only if:

(Def. 1) I_1 is lower-bounded and for every chain L of I_1 such that L is non empty holds sup L exists in I_1.

One can prove the following proposition

(1) Let P_1, P_2 be non empty posets, K be a non empty chain of P_1, and h be a monotone function from P_1 into P_2. Then $h^\circ K$ is a non empty chain of P_2.

Let us note that there exists a poset which is strict, chain-complete, and non empty.

Let us mention that every relational structure which is chain-complete is also lower-bounded.

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For simplicity, we adopt the following rules: x, y denote sets, P, Q denote strict chain-complete non empty posets, L denotes a non empty chain of P, M denotes a non empty chain of Q, p denotes an element of P, f denotes a monotone function from P into Q, and g, g_1, g_2 denote monotone functions from P into P.

We now state the proposition

(2) $\sup(f^\circ L) \leq f(\sup L)$.

2. Fixpoint Theorem for Continuous Functions on Chain-complete Posets

Let P be a non empty poset, let g be a monotone function from P into P, and let p be an element of P. The functor $\mathrm{iterSet}(g, p)$ yields a non empty set and is defined by:

(Def. 2) $\mathrm{iterSet}(g, p) = \{x \in P\colon \bigvee_{n\,:\,\text{natural number}}\ x = g^n(p)\}$.

Next we state the proposition

(3) $\mathrm{iterSet}(g, \perp_P)$ is a non empty chain of P.

Let us consider P and let g be a monotone function from P into P. The functor $\text{iter-min}\, g$ yields a non empty chain of P and is defined by:

(Def. 3) $\text{iter-min}\, g = \mathrm{iterSet}(g, \perp_P)$.

The following propositions are true:

(4) $\sup \text{iter-min}\, g = \sup(g^\circ\, \text{iter-min}\, g)$.

(5) If $g_1 \leq g_2$, then $\sup \text{iter-min}\, g_1 \leq \sup \text{iter-min}\, g_2$.

Let P, Q be non empty posets and let f be a function from P into Q. We say that f is continuous if and only if:

(Def. 4) f is monotone and for every non empty chain L of P holds f preserves sup of L.

We now state two propositions:

(6) For every function f from P into Q holds f is continuous iff f is monotone and for every L holds $f(\sup L) = \sup(f^\circ L)$.

(7) For every element z of Q holds $P \longmapsto z$ is continuous.

Let us consider P, Q. Observe that there exists a function from P into Q which is continuous.

Let us consider P, Q. One can verify that every function from P into Q which is continuous is also monotone.

The following proposition is true

(8) For every monotone function f from P into Q such that for every L holds $f(\sup L) \leq \sup(f^\circ L)$ holds f is continuous.

Let us consider P and let g be a monotone function from P into P. Let us assume that g is continuous. The least fixpoint of g yields an element of P and is defined by the conditions (Def. 5).

(Def. 5)(i) The least fixpoint of g is a fixpoint of g, and

(ii) for every p such that p is a fixpoint of g holds the least fixpoint of $g \leq p$.

One can prove the following propositions:

(9) For every continuous function g from P into P holds the least fixpoint of $g = \sup \text{iter-min}\, g$.

(10) Let g_1, g_2 be continuous functions from P into P. If $g_1 \leq g_2$, then the least fixpoint of $g_1 \leq$ the least fixpoint of g_2.

3. Function Space of Continuous Functions on Chain-complete Posets

Let us consider P, Q. The functor ConFuncs(P, Q) yields a non empty set and is defined by the condition (Def. 6).

(Def. 6) ConFuncs$(P, Q) = \{x; x$ ranges over elements of (the carrier of $Q)^{\text{the carrier of } P}$: $\bigvee_{f\,:\ \text{continuous function from } P \text{ into } Q}\ f = x\}$.

Let us consider P, Q. The functor ConRelat(P, Q) yielding a binary relation on ConFuncs(P, Q) is defined by the condition (Def. 7).

(Def. 7) Let given x, y. Then $\langle x, y \rangle \in$ ConRelat(P, Q) if and only if the following conditions are satisfied:

(i) $x \in$ ConFuncs(P, Q),

(ii) $y \in$ ConFuncs(P, Q), and

(iii) there exist functions f, g from P into Q such that $x = f$ and $y = g$ and $f \leq g$.

Let us consider P, Q. One can verify the following observations:

* ConRelat(P, Q) is reflexive,
* ConRelat(P, Q) is transitive, and
* ConRelat(P, Q) is antisymmetric.

Let us consider P, Q. The functor ConPoset(P, Q) yielding a strict non empty poset is defined as follows:

(Def. 8) ConPoset$(P, Q) = \langle$ConFuncs(P, Q), ConRelat$(P, Q)\rangle$.

In the sequel F is a non empty chain of ConPoset(P, Q).

Let us consider P, Q, F, p. The functor F-image(p) yielding a non empty chain of Q is defined as follows:

(Def. 9) F-image$(p) = \{x \in Q$: $\bigvee_{f\,:\ \text{continuous function from } P \text{ into } Q} (f \in F \wedge x = f(p))\}$.

Let us consider P, Q, F. The functor sup-func F yields a function from P into Q and is defined as follows:

(Def. 10) For all p, M such that $M = F$-image(p) holds (sup-func $F)(p) = \sup M$.

Let us consider P, Q, F. One can check that sup-func F is continuous.

The following proposition is true

(11) Sup F exists in ConPoset(P,Q) and sup-func $F = \bigsqcup_{\mathrm{ConPoset}(P,Q)} F$.

Let us consider P, Q. The functor min-func(P,Q) yielding a function from P into Q is defined as follows:

(Def. 11) min-func$(P,Q) = P \longmapsto \bot_Q$.

Let us consider P, Q. One can check that min-func(P,Q) is continuous.

The following proposition is true

(12) For every element f of ConPoset(P,Q) such that $f =$ min-func(P,Q) holds $f \leq$ the carrier of ConPoset(P,Q).

Let us consider P, Q. Note that ConPoset(P,Q) is chain-complete.

4. Continuity of Fixpoint Function from ConPoset(P,P) into P

Let us consider P. The functor fix-func P yielding a function from ConPoset(P,P) into P is defined by the condition (Def. 12).

(Def. 12) Let g be an element of ConPoset(P,P) and h be a continuous function from P into P. If $g = h$, then (fix-func $P)(g) =$ the least fixpoint of h.

Let us consider P. One can check that fix-func P is continuous.

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