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Complex Integral[1]

Masahiko Yamazaki
Shinshu University
Nagano, Japan

Hiroshi Yamazaki
Shinshu University
Nagano, Japan

Katsumi Wasaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we defined complex curve and complex integral. Then we have proved the linearity for the complex integral. Furthermore, we have proved complex integral of complex curve's connection is the sum of each complex integral of individual complex curve.

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The terminology and notation used here are introduced in the following articles: [10], [2], [14], [11], [12], [3], [4], [1], [7], [15], [5], [13], [8], [17], [9], [16], and [6].

1. The Definition of Complex Curve and Complex Integral

In this paper t is an element of $\mathbb{R}$.

The function $\mathbb{R}^2 \to \mathbb{C}$ from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{C}$ is defined as follows:

(Def. 1) For every element p of $\mathbb{R} \times \mathbb{R}$ and for all elements a, b of $\mathbb{R}$ such that $a = p_{\mathbf{1}}$ and $b = p_{\mathbf{2}}$ holds $(\mathbb{R}^2 \to \mathbb{C})(\langle a, b \rangle) = a + b \cdot i$.

Let a, b be real numbers, let x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\int(f, x, y, a, b, Z)$ yielding a complex number is defined by the condition (Def. 2).

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(Def. 2) There exist partial functions u_0, v_0 from $\mathbb{R}$ to $\mathbb{R}$ such that $u_0 = \Re(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $v_0 = \Im(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $\int(f, x, y, a, b, Z) = \int\limits_a^b (u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z})(x) dx + \int\limits_a^b (v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z})(x) dx \cdot i$.

Let C be a partial function from $\mathbb{R}$ to $\mathbb{C}$. We say that C is C_1-curve-like if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers a, b and there exist partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset Z of $\mathbb{R}$ such that
$a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$.

Let us observe that there exists a partial function from $\mathbb{R}$ to $\mathbb{C}$ which is C_1-curve-like.

A C_1-curve is a C_1-curve-like partial function from $\mathbb{R}$ to $\mathbb{C}$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. Let us assume that $\operatorname{rng} C \subseteq \operatorname{dom} f$. The functor $\int\limits_C f(x) dx$ yields a complex number and is defined by the condition (Def. 4).

(Def. 4) There exist real numbers a, b and there exist partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset Z of $\mathbb{R}$ such that
$a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$ and $\int\limits_C f(x) dx = \int(f, x, y, a, b, Z)$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. We say that f is integrable on C if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let a, b be real numbers, x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and u_0, v_0 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$. Then $u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z}$ is integrable on $[a, b]$ and $v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z}$ is integrable on $[a, b]$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. We say that f is bounded on C if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let a, b be real numbers, x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and u_0, v_0 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\,y)\restriction[a, b]$. Then $(u_0\,x'_{\restriction Z} - v_0\,y'_{\restriction Z})\restriction[a, b]$ is bounded and $(v_0\,x'_{\restriction Z} + u_0\,y'_{\restriction Z})\restriction[a, b]$ is bounded.

2. Linearity of Complex Intergal

Next we state two propositions:

(1) Let f, g be partial functions from $\mathbb{C}$ to $\mathbb{C}$ and C be a C_1-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and $\operatorname{rng} C \subseteq \operatorname{dom} g$ and f is integrable on C and g is integrable on C and f is bounded on C and g is bounded on C. Then $\int\limits_C (f + g)(x)dx = \int\limits_C f(x)dx + \int\limits_C g(x)dx$.

(2) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and C be a C_1-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and f is integrable on C and f is bounded on C. Let r be a real number. Then $\int\limits_C (r\,f)(x)dx = r \cdot \int\limits_C f(x)dx$.

3. Complex Integral of Complex Curve's Connection

We now state the proposition

(3) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, C, C_1, C_2 be C_1-curves, and a, b, d be real numbers. Suppose that $\operatorname{rng} C \subseteq \operatorname{dom} f$ and f is integrable on C and f is bounded on C and $a \leq b$ and $\operatorname{dom} C = [a, b]$ and $d \in [a, b]$ and $\operatorname{dom} C_1 = [a, d]$ and $\operatorname{dom} C_2 = [d, b]$ and for every t such that $t \in \operatorname{dom} C_1$ holds $C(t) = C_1(t)$ and for every t such that $t \in \operatorname{dom} C_2$ holds $C(t) = C_2(t)$. Then $\int\limits_C f(x)dx = \int\limits_{C_1} f(x)dx + \int\limits_{C_2} f(x)dx$.

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