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Dilworth's Decomposition Theorem for Posets[1]

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Summary. The following theorem is due to Dilworth [8]: Let P be a partially ordered set. If the maximal number of elements in an independent subset (anti-chain) of P is k, then P is the union of k chains (cliques).

In this article we formalize an elegant proof of the above theorem for finite posets by Perles [13]. The result is then used in proving the case of infinite posets following the original proof of Dilworth [8].

A dual of Dilworth's theorem also holds: a poset with maximum clique m is a union of m independent sets. The proof of this dual fact is considerably easier; we follow the proof by Mirsky [11]. Mirsky states also a corollary that a poset of $r \times s + 1$ elements possesses a clique of size $r + 1$ or an independent set of size $s + 1$, or both. This corollary is then used to prove the result of Erdős and Szekeres [9].

Instead of using posets, we drop reflexivity and state the facts about anti-symmetric and transitive relations.

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The articles [1], [15], [14], [7], [2], [16], [3], [12], [17], [5], [10], [4], and [6] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *FraenkelFinCard1* deals with a finite non empty set $\mathcal{A}$, a finite set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

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$\overline{\overline{\mathcal{B}}} \leq \overline{\overline{\mathcal{A}}}$

provided the following condition is satisfied:

- $\mathcal{B} = \{\mathcal{F}(w); w$ ranges over elements of $\mathcal{A} : \mathcal{P}[w]\}$.

Next we state the proposition

(1) For all sets X, Y, x such that $x \notin X$ holds $X \setminus (Y \cup \{x\}) = X \setminus Y$.

Let us note that every set which is empty is also $\subseteq$-linear and there exists a set which is empty and $\subseteq$-linear.

Let X be a $\subseteq$-linear set. Note that every subset of X is $\subseteq$-linear.

One can prove the following four propositions:

(2) Let X, Y be sets, F be a family of subsets of X, and G be a family of subsets of Y. Then $F \cup G$ is a family of subsets of $X \cup Y$.

(3) Let X, Y be sets, F be a partition of X, and G be a partition of Y. If X misses Y, then $F \cup G$ is a partition of $X \cup Y$.

(4) For all sets X, Y and for every partition F of Y such that $Y \subset X$ holds $F \cup \{X \setminus Y\}$ is a partition of X.

(5) For every infinite set X and for every natural number n there exists a finite subset Y of X such that $\overline{\overline{Y}} > n$.

2. Cliques and Stable Sets

Let R be a relational structure and let S be a subset of R. We say that S is connected if and only if:

(Def. 1) The internal relation of R is connected in S.

Let R be a relational structure and let S be a subset of R. We introduce S is a clique as a synonym of S is connected.

Let R be a relational structure. Note that every subset of R which is trivial is also a clique.

Let R be a relational structure. One can check that there exists a subset of R which is a clique.

Let R be a relational structure. A clique of R is a clique subset of R.

We now state the proposition

(6) Let R be a relational structure and S be a subset of R. Then S is a clique of R if and only if for all elements a, b of R such that $a, b \in S$ and $a \neq b$ holds $a \leq b$ or $b \leq a$.

Let R be a relational structure. Observe that there exists a clique of R which is finite.

Let R be a reflexive relational structure. One can check that every subset of R which is connected is also strongly connected.

Let R be a non empty relational structure. Observe that there exists a clique of R which is finite and non empty.

One can prove the following propositions:

(7) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a clique of R, then $a_1 \leq a_2$ or $a_2 \leq a_1$.

(8) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \leq a_2$ or $a_2 \leq a_1$, then $\{a_1, a_2\}$ is a clique of R.

(9) For every relational structure R and for every clique C of R holds every subset of C is a clique of R.

(10) Let R be a relational structure, C be a finite clique of R, and n be a natural number. If $n \leq \overline{\overline{C}}$, then there exists a finite clique B of R such that $\overline{\overline{B}} = n$.

(11) Let R be a transitive relational structure, C be a clique of R, and x, y be elements of R. If x is maximal in C and $x \leq y$, then $C \cup \{y\}$ is a clique of R.

(12) Let R be a transitive relational structure, C be a clique of R, and x, y be elements of R. If x is minimal in C and $y \leq x$, then $C \cup \{y\}$ is a clique of R.

Let R be a relational structure and let S be a subset of R. We say that S is stable if and only if:

(Def. 2) For all elements x, y of R such that $x, y \in S$ and $x \neq y$ holds $x \not\leq y$ and $y \not\leq x$.

Let R be a relational structure. One can check that every subset of R which is trivial is also stable. Let R be a relational structure. Note that there exists a subset of R which is stable.

Let R be a relational structure. A stable set of R is a stable subset of R.

Let R be a relational structure. Note that there exists a stable set of R which is finite.

Let R be a non empty relational structure. Observe that there exists a stable set of R which is finite and non empty.

The following propositions are true:

(13) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a stable set of R, then $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$.

(14) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, then $\{a_1, a_2\}$ is a stable set of R.

(15) Let R be a relational structure, C be a clique of R, A be a stable set of R, and a, b be sets. If a, $b \in A$ and a, $b \in C$, then $a = b$.

(16) For every relational structure R and for every stable set A of R holds every subset of A is a stable set of R.

(17) Let R be a relational structure, A be a finite stable set of R, and n be a natural number. If $n \leq \overline{\overline{A}}$, then there exists a finite stable set B of R such that $\overline{\overline{B}} = n$.

3. Clique Number and Stability Number

Let R be a relational structure. We say that R has finite clique number if and only if:

(Def. 3) There exists a finite clique C of R such that for every finite clique D of R holds $\overline{\overline{D}} \leq \overline{\overline{C}}$.

Let us observe that every relational structure which is finite has also finite clique number and there exists a relational structure which is non empty, antisymmetric, and transitive and has finite clique number.

Let R be a relational structure with finite clique number. Observe that every clique of R is finite.

Let R be a relational structure with finite clique number. The functor $\omega(R)$ yields a natural number and is defined as follows:

(Def. 4) There exists a finite clique C of R such that $\overline{\overline{C}} = \omega(R)$ and for every finite clique T of R holds $\overline{\overline{T}} \leq \omega(R)$.

Let R be an empty relational structure. Note that $\omega(R)$ is empty.

Let R be a non empty relational structure with finite clique number. Observe that $\omega(R)$ is positive.

Next we state two propositions:

(18) For every non empty relational structure R with finite clique number such that Ω_R is a stable set of R holds $\omega(R) = 1$.

(19) For every relational structure R with finite clique number such that $\omega(R) = 1$ holds Ω_R is a stable set of R.

Let R be a relational structure. We say that R has finite stability number if and only if:

(Def. 5) There exists a finite stable set A of R such that for every finite stable set B of R holds $\overline{\overline{B}} \leq \overline{\overline{A}}$.

One can verify that every relational structure which is finite has also finite stability number and there exists a relational structure which is antisymmetric, transitive, and non empty and has finite stability number.

Let R be a relational structure with finite stability number. Note that every stable set of R is finite.

Let R be a relational structure with finite stability number. The functor $\alpha(R)$ yielding a natural number is defined by:

(Def. 6) There exists a finite stable set A of R such that $\overline{\overline{A}} = \alpha(R)$ and for every finite stable set T of R holds $\overline{\overline{T}} \leq \alpha(R)$.

Let R be an empty relational structure. Observe that $\alpha(R)$ is empty.

Let R be a non empty relational structure with finite stability number. One can verify that $\alpha(R)$ is positive.

We now state two propositions:

(20) For every non empty relational structure R with finite stability number such that Ω_R is a clique of R holds $\alpha(R) = 1$.

(21) For every relational structure R with finite stability number such that $\alpha(R) = 1$ holds Ω_R is a clique of R.

Let us mention that every relational structure which has finite clique number and finite stability number is also finite.

4. Lower and Upper Sets, Minimal and Maximal Elements

Let R be a relational structure and let X be a subset of R. The functor $\mathrm{Lower}\, X$ yields a subset of R and is defined by:

(Def. 7) $\mathrm{Lower}\, X = X \cup {\downarrow}X$.

The functor $\mathrm{Upper}\, X$ yielding a subset of R is defined as follows:

(Def. 8) $\mathrm{Upper}\, X = X \cup {\uparrow}X$.

One can prove the following propositions:

(22) Let R be an antisymmetric transitive relational structure, A be a stable set of R, and z be a set. If $z \in \mathrm{Upper}\, A$ and $z \in \mathrm{Lower}\, A$, then $z \in A$.

(23) Let R be a relational structure with finite stability number and A be a stable set of R. If $\overline{\overline{A}} = \alpha(R)$, then $\mathrm{Upper}\, A \cup \mathrm{Lower}\, A = \Omega_R$.

(24) Let R be a transitive relational structure, x be an element of R, and S be a subset of R. If x is minimal in $\mathrm{Lower}\, S$, then x is minimal in Ω_R.

(25) Let R be a transitive relational structure, x be an element of R, and S be a subset of R. If x is maximal in $\mathrm{Upper}\, S$, then x is maximal in Ω_R.

Let R be a relational structure. The functor $\mathrm{minimals}(R)$ yielding a subset of R is defined as follows:

(Def. 9)(i) For every element x of R holds $x \in \mathrm{minimals}(R)$ iff x is minimal in Ω_R if R is non empty,

(ii) $\mathrm{minimals}(R) = \emptyset$, otherwise.

The functor $\mathrm{maximals}(R)$ yielding a subset of R is defined as follows:

(Def. 10)(i) For every element x of R holds $x \in \mathrm{maximals}(R)$ iff x is maximal in Ω_R if R is non empty,

(ii) $\mathrm{maximals}(R) = \emptyset$, otherwise.

Let R be a non empty antisymmetric transitive relational structure with finite clique number. One can verify that $\mathrm{maximals}(R)$ is non empty and $\mathrm{minimals}(R)$ is non empty.

Let R be a relational structure. Note that $\mathrm{minimals}(R)$ is stable and $\mathrm{maximals}(R)$ is stable.

The following two propositions are true:

(26) For every relational structure R and for every stable set A of R such that minimals$(R) \not\subseteq A$ holds minimals$(R) \not\subseteq \mathrm{Upper}\, A$.

(27) For every relational structure R and for every stable set A of R such that maximals$(R) \not\subseteq A$ holds maximals$(R) \not\subseteq \mathrm{Lower}\, A$.

5. Substructures

Let R be a relational structure and let X be a finite subset of R. Observe that sub(X) is finite.

One can prove the following propositions:

(28) For every relational structure R and for every subset S of R holds every clique of sub(S) is a clique of R.

(29) Let R be a relational structure, S be a subset of R, and C be a clique of R. Then $C \cap S$ is a clique of sub(S).

(30) For every relational structure R and for every subset S of R holds every stable set of sub(S) is a stable set of R.

(31) Let R be a relational structure, S be a subset of R, and A be a stable set of R. Then $A \cap S$ is a stable set of sub(S).

(32) Let R be a relational structure, S be a subset of R, B be a subset of sub(S), x be an element of sub(S), and y be an element of R. If $x = y$ and x is maximal in B, then y is maximal in B.

(33) Let R be a relational structure, S be a subset of R, B be a subset of sub(S), x be an element of sub(S), and y be an element of R. If $x = y$ and x is minimal in B, then y is minimal in B.

(34) Let R be a transitive relational structure, A be a stable set of R, C be a clique of sub$(\mathrm{Lower}\, A)$, and a, b be elements of R. If $a \in A$ and a, $b \in C$, then $a = b$ or $b \leq a$.

(35) Let R be a transitive relational structure, A be a stable set of R, C be a clique of sub$(\mathrm{Upper}\, A)$, and a, b be elements of R. If $a \in A$ and a, $b \in C$, then $a = b$ or $a \leq b$.

Let R be a relational structure with finite clique number and let S be a subset of R. One can verify that sub(S) has finite clique number.

Let R be a relational structure with finite stability number and let S be a subset of R. One can verify that sub(S) has finite stability number.

The following propositions are true:

(36) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R. Then there exists an element y of R such that y is minimal in Ω_R but $y = x$ or $y < x$.

(37) For every antisymmetric transitive relational structure R with finite clique number holds Upper minimals$(R) = \Omega_R$.

(38) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R. Then there exists an element y of R such that y is maximal in Ω_R but $y = x$ or $x < y$.

(39) For every antisymmetric transitive relational structure R with finite clique number holds Lower maximals$(R) = \Omega_R$.

(40) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R. If minimals$(R) \subseteq A$, then $A =$ minimals(R).

(41) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R. If maximals$(R) \subseteq A$, then $A =$ maximals(R).

(42) For every relational structure R with finite clique number and for every subset S of R holds $\omega(\mathrm{sub}(S)) \leq \omega(R)$.

(43) Let R be a relational structure with finite clique number, C be a clique of R, and S be a subset of R. If $\overline{\overline{C}} = \omega(R)$ and $C \subseteq S$, then $\omega(\mathrm{sub}(S)) = \omega(R)$.

(44) For every relational structure R with finite stability number and for every subset S of R holds $\alpha(\mathrm{sub}(S)) \leq \alpha(R)$.

(45) Let R be a relational structure with finite stability number, A be a stable set of R, and S be a subset of R. If $\overline{\overline{A}} = \alpha(R)$ and $A \subseteq S$, then $\alpha(\mathrm{sub}(S)) = \alpha(R)$.

6. Partitions into Cliques and Stable Sets

Let R be a relational structure and let P be a partition of the carrier of R. We say that P is clique-wise if and only if:

(Def. 11) For every set x such that $x \in P$ holds x is a clique of R.

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is clique-wise.

Let R be a relational structure. A clique-partition of R is a clique-wise partition of the carrier of R.

Let R be an empty relational structure. One can verify that every partition of the carrier of R which is empty is also clique-wise.

Next we state four propositions:

(46) For every finite relational structure R and for every clique-partition C of R holds $\overline{\overline{C}} \geq \alpha(R)$.

(47) Let R be a relational structure with finite stability number, A be a stable set of R, and C be a clique-partition of R. Suppose $\mathrm{Card}\, C = \mathrm{Card}\, A$. Then there exists a function f from A into C such that f is bijective and for every set x such that $x \in A$ holds $x \in f(x)$.

(48) Let R be a finite relational structure, A be a stable set of R, and C be a clique-partition of R. Suppose $\overline{\overline{C}} = \overline{\overline{A}}$. Let c be a set. If $c \in C$, then there exists an element a of A such that $c \cap A = \{a\}$.

(49) Let R be an antisymmetric transitive non empty relational structure with finite stability number, A be a stable set of R, U be a clique-partition of $\mathrm{sub}(\mathrm{Upper}\,A)$, and L be a clique-partition of $\mathrm{sub}(\mathrm{Lower}\,A)$. Suppose $\overline{\overline{A}} = \alpha(R)$ and $\mathrm{Card}\,U = \alpha(R)$ and $\mathrm{Card}\,L = \alpha(R)$. Then there exists a clique-partition C of R such that $\mathrm{Card}\,C = \alpha(R)$.

Let R be a relational structure and let P be a partition of the carrier of R. We say that P is stable-wise if and only if:

(Def. 12) For every set x such that $x \in P$ holds x is a stable set of R.

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is stable-wise.

Let R be a relational structure. A coloring of R is a stable-wise partition of the carrier of R.

Let R be an empty relational structure. Note that every partition of the carrier of R is stable-wise.

We now state the proposition

(50) For every finite relational structure R and for every coloring C of R holds $\overline{\overline{C}} \geq \omega(R)$.

7. Dilworth's Theorem and a Dual

Next we state the proposition

(51) Let R be a finite antisymmetric transitive relational structure. Then there exists a clique-partition C of R such that $\overline{\overline{C}} = \alpha(R)$.

Let R be a non empty relational structure with finite stability number and let C be a subset of R. We say that C is strong-chain if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let S be a finite non empty subset of R. Then there exists a clique-partition P of $\mathrm{sub}(S)$ such that $\overline{\overline{P}} \leq \alpha(R)$ and there exists a set c such that $c \in P$ and $S \cap C \subseteq c$ and for every set d such that $d \in P$ and $d \neq c$ holds $C \cap d = \emptyset$.

Let R be a non empty relational structure with finite stability number. Note that every subset of R which is strong-chain is also a clique.

Let R be an antisymmetric transitive non empty relational structure with finite stability number. Observe that every subset of R which is trivial and non empty is also strong-chain.

The following propositions are true:

(52) Let R be a non empty antisymmetric transitive relational structure with finite stability number. Then there exists a non empty subset S of R such that S is strong-chain and it is not true that there exists a subset D of R such that D is strong-chain and $S \subset D$.

(53) Let R be an antisymmetric transitive relational structure with finite stability number. Then there exists a clique-partition C of R such that $\operatorname{Card} C = \alpha(R)$.

(54) Let R be an antisymmetric transitive relational structure with finite clique number. Then there exists a coloring A of R such that $\operatorname{Card} A = \omega(R)$.

8. Erdős-Szekeres Theorem

One can prove the following two propositions:

(55) Let R be a finite antisymmetric transitive relational structure and r, s be natural numbers. Suppose $\operatorname{Card} R = r \cdot s + 1$. Then there exists a clique C of R such that $\overline{\overline{C}} \geq r + 1$ or there exists a stable set A of R such that $\overline{\overline{A}} \geq s + 1$.

(56) Let f be a real-valued finite sequence and n be a natural number. Suppose $\overline{\overline{f}} = n^{\mathbf{2}} + 1$ and f is one-to-one. Then there exists a real-valued finite subsequence g such that $g \subseteq f$ and $\overline{\overline{g}} \geq n + 1$ and g is increasing or decreasing.

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