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Basic Properties of Even and Odd Functions

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Summary. In this article we present definitions, basic properties and some examples of even and odd functions [6].

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The articles [2], [5], [1], [8], [14], [12], [15], [7], [17], [3], [4], [11], [19], [13], [10], [18], [16], and [9] provide the notation and terminology for this paper.

1. Even and Odd Functions

In this paper x, r denote real numbers.

Let A be a set. We say that A is symmetrical if and only if:

(Def. 1) For every complex number x such that $x \in A$ holds $-x \in A$.

One can check that there exists a subset of $\mathbb{C}$ which is symmetrical.

Let us note that there exists a subset of $\mathbb{R}$ which is symmetrical.

In the sequel A is a symmetrical subset of $\mathbb{C}$.

Let R be a binary relation. We say that R has symmetrical domain if and only if:

(Def. 2) $\operatorname{dom} R$ is symmetrical.

Let us observe that every binary relation which is empty has also symmetrical domain and there exists a binary relation which has symmetrical domain.

Let R be a binary relation with symmetrical domain. One can check that $\operatorname{dom} R$ is symmetrical.

Let X, Y be complex-membered sets and let F be a partial function from X to Y. We say that F is quasi even if and only if:

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(Def. 3) For every x such that x, $-x \in \operatorname{dom} F$ holds $F(-x) = F(x)$.

Let X, Y be complex-membered sets and let F be a partial function from X to Y. We say that F is even if and only if:

(Def. 4) F is quasi even and has symmetrical domain.

Let X, Y be complex-membered sets. Note that every partial function from X to Y which is quasi even and has symmetrical domain is also even and every partial function from X to Y which is even is also quasi even and has symmetrical domain.

Let A be a set, let X, Y be complex-membered sets, and let F be a partial function from X to Y. We say that F is even on A if and only if:

(Def. 5) $A \subseteq \operatorname{dom} F$ and $F{\restriction}A$ is even.

Let X, Y be complex-membered sets and let F be a partial function from X to Y. We say that F is quasi odd if and only if:

(Def. 6) For every x such that x, $-x \in \operatorname{dom} F$ holds $F(-x) = -F(x)$.

Let X, Y be complex-membered sets and let F be a partial function from X to Y. We say that F is odd if and only if:

(Def. 7) F is quasi odd and has symmetrical domain.

Let X, Y be complex-membered sets. Note that every partial function from X to Y which is quasi odd and has symmetrical domain is also odd and every partial function from X to Y which is odd is also quasi odd and has symmetrical domain.

Let A be a set, let X, Y be complex-membered sets, and let F be a partial function from X to Y. We say that F is odd on A if and only if:

(Def. 8) $A \subseteq \operatorname{dom} F$ and $F{\restriction}A$ is odd.

In the sequel F, G denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(1) F is odd on A iff $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $F(x) + F(-x) = 0$.

(2) F is even on A iff $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $F(x) - F(-x) = 0$.

(3) If F is odd on A and for every x such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$.

(4) If $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = -1$, then F is odd on A.

(5) If F is even on A and for every x such that $x \in A$ holds $F(x) \neq 0$, then $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$.

(6) If $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $\frac{F(x)}{F(-x)} = 1$, then F is even on A.

(7) If F is even on A and odd on A, then for every x such that $x \in A$ holds $F(x) = 0$.

(8) If F is even on A, then for every x such that $x \in A$ holds $F(x) = F(|x|)$.

(9) If $A \subseteq \operatorname{dom} F$ and for every x such that $x \in A$ holds $F(x) = F(|x|)$, then F is even on A.

(10) If F is odd on A and G is odd on A, then $F + G$ is odd on A.

(11) If F is even on A and G is even on A, then $F + G$ is even on A.

(12) If F is odd on A and G is odd on A, then $F - G$ is odd on A.

(13) If F is even on A and G is even on A, then $F - G$ is even on A.

(14) If F is odd on A, then $r\,F$ is odd on A.

(15) If F is even on A, then $r\,F$ is even on A.

(16) If F is odd on A, then $-F$ is odd on A.

(17) If F is even on A, then $-F$ is even on A.

(18) If F is odd on A, then F^{-1} is odd on A.

(19) If F is even on A, then F^{-1} is even on A.

(20) If F is odd on A, then $|F|$ is even on A.

(21) If F is even on A, then $|F|$ is even on A.

(22) If F is odd on A and G is odd on A, then $F\,G$ is even on A.

(23) If F is even on A and G is even on A, then $F\,G$ is even on A.

(24) If F is even on A and G is odd on A, then $F\,G$ is odd on A.

(25) If F is even on A, then $r + F$ is even on A.

(26) If F is even on A, then $F - r$ is even on A.

(27) If F is even on A, then $F^{\mathbf{2}}$ is even on A.

(28) If F is odd on A, then $F^{\mathbf{2}}$ is even on A.

(29) If F is odd on A and G is odd on A, then F/G is even on A.

(30) If F is even on A and G is even on A, then F/G is even on A.

(31) If F is odd on A and G is even on A, then F/G is odd on A.

(32) If F is even on A and G is odd on A, then F/G is odd on A.

(33) If F is odd, then $-F$ is odd.

(34) If F is even, then $-F$ is even.

(35) If F is odd, then F^{-1} is odd.

(36) If F is even, then F^{-1} is even.

(37) If F is odd, then $|F|$ is even.

(38) If F is even, then $|F|$ is even.

(39) If F is odd, then $F^{\mathbf{2}}$ is even.

(40) If F is even, then $F^{\mathbf{2}}$ is even.

(41) If F is even, then $r + F$ is even.

(42) If F is even, then $F - r$ is even.

(43) If F is odd, then $r\,F$ is odd.

(44) If F is even, then $r\,F$ is even.

(45) If F is odd and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F + G$ is odd.

(46) If F is even and G is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F + G$ is even.

(47) If F is odd and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F - G$ is odd.

(48) If F is even and G is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F - G$ is even.

(49) If F is odd and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is even.

(50) If F is even and G is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is even.

(51) If F is even and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then $F\,G$ is odd.

(52) If F is odd and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then F/G is even.

(53) If F is even and G is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then F/G is even.

(54) If F is odd and G is even and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then F/G is odd.

(55) If F is even and G is odd and $\operatorname{dom} F \cap \operatorname{dom} G$ is symmetrical, then F/G is odd.

2. Some Examples

The function signum from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 9) For every real number x holds $\mathrm{signum}(x) = \operatorname{sgn} x$.

Let x be a real number. One can verify that $\mathrm{signum}(x)$ is real.

Next we state a number of propositions:

(56) For every real number x such that $x > 0$ holds $\mathrm{signum}(x) = 1$.

(57) For every real number x such that $x < 0$ holds $\mathrm{signum}(x) = -1$.

(58) $\mathrm{signum}(0) = 0$.

(59) For every real number x holds $\mathrm{signum}(-x) = -\mathrm{signum}(x)$.

(60) For every symmetrical subset A of $\mathbb{R}$ holds signum is odd on A.

(61) For every real number x such that $x \geq 0$ holds $|\square|_{\mathbb{R}}(x) = x$.

(62) For every real number x such that $x < 0$ holds $|\square|_{\mathbb{R}}(x) = -x$.

(63) For every real number x holds $|\square|_{\mathbb{R}}(-x) = |\square|_{\mathbb{R}}(x)$.

(64) For every symmetrical subset A of $\mathbb{R}$ holds $|\square|_{\mathbb{R}}$ is even on A.

(65) For every symmetrical subset A of $\mathbb{R}$ holds the function sin is odd on A.

(66) For every symmetrical subset A of $\mathbb{R}$ holds the function cos is even on A.

Let us observe that the function sin is odd.

Let us observe that the function cos is even.

We now state two propositions:

(67) For every symmetrical subset A of $\mathbb{R}$ holds the function sinh is odd on A.

(68) For every symmetrical subset A of $\mathbb{R}$ holds the function cosh is even on A.

Let us note that the function sinh is odd.

Let us mention that the function cosh is even.

The following propositions are true:

(69) If $A \subseteq]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function tan is odd on A.

(70) Suppose $A \subseteq \mathrm{dom}\,(\text{the function tan})$ and for every x such that $x \in A$ holds $(\text{the function cos})(x) \neq 0$. Then the function tan is odd on A.

(71) Suppose $A \subseteq \mathrm{dom}\,(\text{the function cot})$ and for every x such that $x \in A$ holds $(\text{the function sin})(x) \neq 0$. Then the function cot is odd on A.

(72) If $A \subseteq [-1, 1]$, then the function arctan is odd on A.

(73) For every symmetrical subset A of $\mathbb{R}$ holds |the function sin | is even on A.

(74) For every symmetrical subset A of $\mathbb{R}$ holds |the function cos | is even on A.

(75) For every symmetrical subset A of $\mathbb{R}$ holds (the function sin) $^{-1}$ is odd on A.

(76) For every symmetrical subset A of $\mathbb{R}$ holds (the function cos) $^{-1}$ is even on A.

(77) For every symmetrical subset A of $\mathbb{R}$ holds $-$the function sin is odd on A.

(78) For every symmetrical subset A of $\mathbb{R}$ holds $-$the function cos is even on A.

(79) For every symmetrical subset A of $\mathbb{R}$ holds (the function sin)$^{\mathbf{2}}$ is even on A.

(80) For every symmetrical subset A of $\mathbb{R}$ holds (the function cos)$^{\mathbf{2}}$ is even on A.

In the sequel B denotes a symmetrical subset of $\mathbb{R}$.

One can prove the following propositions:

(81) If $B \subseteq \mathrm{dom}$ (the function sec), then the function sec is even on B.

(82) If for every real number x such that $x \in B$ holds (the function cos)$(x) \neq 0$, then the function sec is even on B.

(83) If $B \subseteq \mathrm{dom}$ (the function cosec), then the function cosec is odd on B.

(84) If for every real number x such that $x \in B$ holds (the function sin)$(x) \neq 0$, then the function cosec is odd on B.

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