FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 23–35, **2009**
DOI: 10.2478/v10037-009-0003-0

Several Integrability Formulas of Special Functions. Part II

Bo Li
Qingdao University of Science and Technology
China

Yanping Zhuang
Qingdao University of Science and Technology
China

Yanhong Men
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including the trigonometric function, the hyperbolic function and the polynomial function [3].

MML identifier: `INTEGR11`, version: `7.11.01 4.117.1046`

The articles [10], [23], [19], [21], [22], [1], [8], [15], [9], [2], [4], [17], [5], [13], [16], [14], [18], [7], [12], [20], [6], and [11] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following rules: r, x, a, b denote real numbers, n, m denote elements of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1)(i) $(\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\cdot(2\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\sin x)^{\mathbf{2}}$.

ISSN 1426–2630(p), 1898-9934(e)

(2)(i) $(\frac{1}{2}\square+0) + \frac{1}{4}$ ((the function sin) $\cdot(2\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) + \frac{1}{4}$ ((the function sin) $\cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\cos x)^{\mathbf{2}}$.

(3) $\frac{1}{n+1}$ $((\square^{n+1})\cdot$ (the function sin)) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}$ (the function sin)$^{n+1})'_{\restriction\mathbb{R}}(x) = (\sin x)^n \cdot \cos x$.

(4)(i) $(-\frac{1}{n+1})$ $((\square^{n+1})\cdot$ (the function cos)) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((-\frac{1}{n+1})$ (the function cos)$^{n+1})'_{\restriction\mathbb{R}}(x) = (\cos x)^n \cdot \sin x$.

(5) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then

(i) $\frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \cos(m\cdot x)\cdot\cos(n\cdot x)$.

(6) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then

(i) $\frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m\cdot x)\cdot\sin(n\cdot x)$.

(7) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then

(i) $-\frac{1}{2\cdot(m+n)}$ ((the function cos) $\cdot((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}$ ((the function cos) $\cdot((m-n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(-\frac{1}{2\cdot(m+n)}$ ((the function cos) $\cdot((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}$ ((the function cos) $\cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m\cdot x)\cdot\cos(n\cdot x)$.

(8) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}$ ((the function sin) $\cdot(n\square+0)) - (\frac{1}{n}\square+0)$ ((the function cos) $\cdot(n\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}$ ((the function sin) $\cdot(n\square+0)) - (\frac{1}{n}\square+0)$ ((the function cos) $\cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x\cdot\sin(n\cdot x)$.

(9) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}$ ((the function cos) $\cdot(n\square+0)) + (\frac{1}{n}\square+0)$ ((the function sin) $\cdot(n\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}$ ((the function cos) $\cdot(n\square+0)) + (\frac{1}{n}\square+0)$ ((the function sin) $\cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x\cdot\cos(n\cdot x)$.

(10)(i) $(1\square+0)$ (the function cosh)$-$the function sinh is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square+0)$ (the function cosh)$-$the function sinh$)'_{\restriction\mathbb{R}}(x) = x\cdot\sinh x$.

(11)(i) $(1\square+0)$ (the function sinh)$-$the function cosh is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square{+}0)\,(\text{the function sinh}) - \text{the function cosh})'_{\restriction\mathbb{R}}(x) = x \cdot \cosh x$.

(12) If $a \cdot (n+1) \neq 0$, then $\frac{1}{a\cdot(n+1)}\,(a\square{+}b)^{n+1}$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{a\cdot(n+1)}\,(a\square{+}b)^{n+1})'_{\restriction\mathbb{R}}(x) = (a \cdot x + b)^n$.

2. Integrability Formulas

Next we state a number of propositions:

(13) $\int\limits_A (\text{the function sin})^{\mathbf{2}}(x)dx = \frac{1}{2} \cdot \sup A - \frac{1}{4} \cdot \sin(2 \cdot \sup A) - (\frac{1}{2} \cdot \inf A - \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(14) $\int\limits_{[0,\pi]} (\text{the function sin})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(15) $\int\limits_{[0,2\cdot\pi]} (\text{the function sin})^{\mathbf{2}}(x)dx = \pi$.

(16) $\int\limits_A (\text{the function cos})^{\mathbf{2}}(x)dx = (\frac{1}{2} \cdot \sup A + \frac{1}{4} \cdot \sin(2 \cdot \sup A)) - (\frac{1}{2} \cdot \inf A + \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(17) $\int\limits_{[0,\pi]} (\text{the function cos})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(18) $\int\limits_{[0,2\cdot\pi]} (\text{the function cos})^{\mathbf{2}}(x)dx = \pi$.

(19) $\int\limits_A ((\text{the function sin})^n\,(\text{the function cos}))(x)dx = \frac{1}{n+1} \cdot (\sin \sup A)^{n+1} - \frac{1}{n+1} \cdot (\sin \inf A)^{n+1}$.

(20) $\int\limits_{[0,\pi]} ((\text{the function sin})^n\,(\text{the function cos}))(x)dx = 0$.

(21) $\int\limits_{[0,2\cdot\pi]} ((\text{the function sin})^n\,(\text{the function cos}))(x)dx = 0$.

(22) $\int\limits_A ((\text{the function cos})^n\,(\text{the function sin}))(x)dx = (-\frac{1}{n+1}) \cdot (\cos \sup A)^{n+1} - (-\frac{1}{n+1}) \cdot (\cos \inf A)^{n+1}$.

(23) $\int\limits_{[0,2\cdot\pi]} ((\text{the function } \cos)^n \, (\text{the function } \sin))(x)dx = 0.$

(24) $\int\limits_{[-\frac{\pi}{2},\frac{\pi}{2}]} ((\text{the function } \cos)^n \, (\text{the function } \sin))(x)dx = 0.$

(25) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int\limits_A (((\text{the function } \cos)\cdot(m\square+0))\,((\text{the function } \cos)\cdot(n\square+0)))(x)dx = (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)) - (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)).$

(26) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int\limits_A (((\text{the function } \sin)\cdot(m\square+0))\,((\text{the function } \sin)\cdot(n\square+0)))(x)dx = \frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A) - (\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)).$

(27) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int\limits_A (((\text{the function } \sin)\cdot(m\square+0))\,((\text{the function } \cos)\cdot(n\square+0)))(x)dx = -\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\sup A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\sup A) - (-\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\inf A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\inf A)).$

(28) If $n \neq 0$, then $\int\limits_A ((1\square+0)\,((\text{the function } \sin)\cdot(n\square+0)))(x)dx = \frac{1}{n^{\mathbf{2}}}\cdot\sin(n\cdot\sup A)-\frac{1}{n}\cdot\sup A\cdot\cos(n\cdot\sup A)-(\frac{1}{n^{\mathbf{2}}}\cdot\sin(n\cdot\inf A)-\frac{1}{n}\cdot\inf A\cdot\cos(n\cdot\inf A)).$

(29) If $n \neq 0$, then $\int\limits_A ((1\square+0)\,((\text{the function } \cos)\cdot(n\square+0)))(x)dx = (\frac{1}{n^{\mathbf{2}}}\cdot\cos(n\cdot\sup A)+\frac{1}{n}\cdot\sup A\cdot\sin(n\cdot\sup A))-(\frac{1}{n^{\mathbf{2}}}\cdot\cos(n\cdot\inf A)+\frac{1}{n}\cdot\inf A\cdot\sin(n\cdot\inf A)).$

(30) $\int\limits_A ((1\square+0)\,(\text{the function } \sinh))(x)dx = \sup A\cdot\cosh\sup A-\sinh\sup A-(\inf A\cdot\cosh\inf A-\sinh\inf A).$

(31) $\int\limits_A ((1\square+0)\,(\text{the function } \cosh))(x)dx = \sup A\cdot\sinh\sup A-\cosh\sup A-(\inf A\cdot\sinh\inf A-\cosh\inf A).$

(32) If $a \cdot (n+1) \neq 0$, then $\int\limits_A (a\Box{+}b)^n(x)dx = \frac{1}{a \cdot (n+1)} \cdot (a \cdot \sup A + b)^{n+1} - \frac{1}{a \cdot (n+1)} \cdot (a \cdot \inf A + b)^{n+1}$.

3. Addenda

In the sequel f, f_1, f_2, f_3, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

(33) If $Z \subseteq \operatorname{dom}(\frac{1}{2}\, f)$ and $f = \Box^2$, then $\frac{1}{2}\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{2}\, f)'_{\restriction Z}(x) = x$.

(34) If $A \subseteq Z = \operatorname{dom}(\frac{1}{2}\, (\Box^2))$, then $\int\limits_A \mathrm{id}_Z(x)dx = \frac{1}{2} \cdot (\sup A)^2 - \frac{1}{2} \cdot (\inf A)^2$.

(35) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $g(x) = x$ and $g(x) \neq 0$ and $f(x) = -\frac{1}{x^2}$ and $Z = \operatorname{dom} g$ and $\operatorname{dom} f = Z$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = (\sup A)^{-1} - (\inf A)^{-1}$.

(36) Suppose that

(i) $A \subseteq Z$,

(ii) $f_1 = \Box^2$,

(iii) for every x such that $x \in Z$ holds $f_2(x) = 1$ and $x \neq 0$ and $f(x) = \frac{2 \cdot x}{(1+x^2)^2}$,

(iv) $\operatorname{dom}(\frac{f_1}{f_2+f_1}) = Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (\frac{f_1}{f_2 + f_1})(\sup A) - (\frac{f_1}{f_2 + f_1})(\inf A)$.

(37) Suppose $Z \subseteq \operatorname{dom}($(the function tan)+(the function sec)$)$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) (the function tan)+(the function sec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)+(the function sec))$'_{\restriction Z}(x) = \frac{1}{1-\sin x}$.

(38) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{1}{1-\sin x}$,

(iii) dom((the function tan)+(the function sec)) $= Z$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (\tan\sup A + \sec\sup A) - (\tan\inf A + \sec\inf A)$.

(39) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan})-(\text{the function sec}))$ and for every x such that $x \in Z$ holds $1+\sin x \neq 0$ and $1-\sin x \neq 0$. Then

(i) (the function tan)$-$(the function sec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)$-$(the function sec))$'_{\restriction Z}(x) = \frac{1}{1+\sin x}$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1+\sin x \neq 0$ and $1-\sin x \neq 0$ and $f(x) = \frac{1}{1+\sin x}$,

(iii) dom((the function tan)$-$(the function sec)) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \tan\sup A - \sec\sup A - (\tan\inf A - \sec\inf A)$.

(41) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} + \text{the function cosec})$ and for every x such that $x \in Z$ holds $1+\cos x \neq 0$ and $1-\cos x \neq 0$. Then

(i) $-$the function cot $+$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $+$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1+\cos x}$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1+\cos x \neq 0$ and $1-\cos x \neq 0$ and $f(x) = \frac{1}{1+\cos x}$,

(iii) dom($-$the function cot $+$ the function cosec) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-\cot\sup A + \mathrm{cosec}\sup A) - (-\cot\inf A + \mathrm{cosec}\inf A)$.

(43) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} - \text{the function cosec})$ and for every x such that $x \in Z$ holds $1+\cos x \neq 0$ and $1-\cos x \neq 0$. Then

(i) $-$the function cot $-$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $-$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1-\cos x}$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1+\cos x \neq 0$ and $1-\cos x \neq 0$ and $f(x) = \frac{1}{1-\cos x}$,

(iii) dom($-$the function cot $-$ the function cosec) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = -\cot \sup A - \operatorname{cosec} \sup A - (-\cot \inf A - \operatorname{cosec} \inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{1+x^2}$,

(iv) dom (the function arctan) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \arctan \sup A - \arctan \inf A$.

(46) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{r}{1+x^2}$,

(iv) dom(r the function arctan) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = r \cdot \arctan \sup A - r \cdot \arctan \inf A$.

(47) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{1+x^2}$,

(iv) dom (the function arccot) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \operatorname{arccot} \sup A - \operatorname{arccot} \inf A$.

(48) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{r}{1+x^2}$,

(iv) dom(r the function arccot) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = r \cdot \operatorname{arccot} \sup A - r \cdot \operatorname{arccot} \inf A$.

(49) Suppose $Z \subseteq$ dom((id$_Z$ + the function cot)$-$the function cosec) and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $(\mathrm{id}_Z + \text{the function cot}) - \text{the function cosec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z + \text{the function cot}) - \text{the function cosec})'_{\restriction Z}(x) = \frac{\cos x}{1+\cos x}$.

(50) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{1+\cos x}$,
(iii) $\mathrm{dom}((\mathrm{id}_Z + \text{the function cot}) - \text{the function cosec}) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A + \cot \sup A) - \operatorname{cosec} \sup A - ((\inf A + \cot \inf A) - \operatorname{cosec} \inf A)$.

(51) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + \text{the function cot} + \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then
(i) $\mathrm{id}_Z + \text{the function cot} + \text{the function cosec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z + \text{the function cot} + \text{the function cosec})'_{\restriction Z}(x) = \frac{\cos x}{\cos x - 1}$.

(52) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{\cos x - 1}$,
(iii) $\mathrm{dom}(\mathrm{id}_Z + \text{the function cot} + \text{the function cosec}) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A + \cot \sup A + \operatorname{cosec} \sup A) - (\inf A + \cot \inf A + \operatorname{cosec} \inf A)$.

(53) Suppose $Z \subseteq \mathrm{dom}((\mathrm{id}_Z - \text{the function tan}) + \text{the function sec})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then
(i) $(\mathrm{id}_Z - \text{the function tan}) + \text{the function sec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z - \text{the function tan}) + \text{the function sec})'_{\restriction Z}(x) = \frac{\sin x}{\sin x + 1}$.

(54) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{1+\sin x}$,
(iii) $Z \subseteq \mathrm{dom}((\mathrm{id}_Z - \text{the function tan}) + \text{the function sec})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\sup A - \tan \sup A) + \sec \sup A) - ((\inf A - \tan \inf A) + \sec \inf A)$.

(55) Suppose $Z \subseteq \operatorname{dom}(\mathrm{id}_Z -$the function tan$-$the function sec) and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) $\mathrm{id}_Z -$ the function tan$-$the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($\mathrm{id}_Z -$ the function tan$-$the function sec$)'_{\restriction Z}(x) = \frac{\sin x}{\sin x - 1}$.

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{\sin x - 1}$,

(iii) $Z \subseteq \operatorname{dom}(\mathrm{id}_Z -$ the function tan$-$the function sec),

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \sup A - \tan \sup A - \sec \sup A - (\inf A - \tan \inf A - \sec \inf A)$.

(57) Suppose $Z \subseteq \operatorname{dom}(($the function tan$)-\mathrm{id}_Z)$. Then (the function tan)$-\mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds ((the function tan)$-\mathrm{id}_Z)'_{\restriction Z}(x) = (\tan x)^{\mathbf{2}}$.

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds (the function cos)$(x) > 0$ and $f(x) = (\tan x)^{\mathbf{2}}$,

(iii) $Z \subseteq \operatorname{dom}(($the function tan$)-\mathrm{id}_Z)$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \tan \sup A - \sup A - (\tan \inf A - \inf A)$.

(59) Suppose $Z \subseteq \operatorname{dom}(-$the function cot $- \mathrm{id}_Z)$. Then $-$the function cot $- \mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $(-$the function cot $- \mathrm{id}_Z)'_{\restriction Z}(x) = (\cot x)^{\mathbf{2}}$.

(60) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and $f(x) = (\cot x)^{\mathbf{2}}$,

(iii) $Z \subseteq \operatorname{dom}(-$the function cot $- \mathrm{id}_Z)$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = -\cot\sup A - \sup A - (-\cot\inf A - \inf A)$.

(61) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\cos x)^{2}}$ and $\cos x \neq 0$ and dom (the function tan) $= Z =$ dom f and $f{\restriction}A$ is continuous. Then $\int\limits_{A} f(x)dx = \tan\sup A - \tan\inf A$.

(62) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\sin x)^{2}}$ and $\sin x \neq 0$ and dom (the function cot) $= Z =$ dom f and $f{\restriction}A$ is continuous. Then $\int\limits_{A} f(x)dx = \cot\sup A - \cot\inf A$.

(63) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x - (\cos x)^{2}}{(\cos x)^{2}}$ and $Z \subseteq$ dom((the function sec)$-\mathrm{id}_{Z}$) and $Z =$ dom f and $f{\restriction}A$ is continuous. Then $\int\limits_{A} f(x)dx = \sec\sup A - \sup A - (\sec\inf A - \inf A)$.

(64) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = \frac{\cos x - (\sin x)^{2}}{(\sin x)^{2}}$,

(iii) $Z \subseteq$ dom($-$the function cosec $-\ \mathrm{id}_{Z}$),

(iv) $Z =$ dom f, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = -\mathrm{cosec}\sup A - \sup A - (-\mathrm{cosec}\inf A - \inf A)$.

The following propositions are true:

(65) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $\sin x > 0$,

(iii) $Z \subseteq$ dom((the function ln) $\cdot$(the function sin)),

(iv) $Z =$ dom (the function cot), and

(v) (the function cot)$\restriction A$ is continuous.

Then $\int\limits_{A}$ (the function cot)$(x)dx = \ln\sin\sup A - \ln\sin\inf A$.

(66) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1,1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{\arcsin x}{\sqrt{1-x^{2}}}$,

(iv) $Z \subseteq$ dom($\frac{1}{2}$ (the function arcsin)^{2}),

(v) $Z =$ dom f, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arcsin \sup A)^2 - \frac{1}{2} \cdot (\arcsin \inf A)^2$.

(67) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{\arccos x}{\sqrt{1-x^2}}$,

(iv) $Z \subseteq \mathrm{dom}(\frac{1}{2}$ (the function arccos)$^2)$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arccos \sup A)^2 - \frac{1}{2} \cdot (\arccos \inf A)^2$.

(68) $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arcsin) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arcsin)$+f^{\frac{1}{2}})$.

(69) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arcsin) $\cdot f_3) = Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ ((the function arcsin) $\cdot f_3) + (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arcsin) $\cdot f_3) \restriction A$ is continuous. Then $\int\limits_A$((the function arcsin) $\cdot f_3)(x)dx =$ $(\sup A \cdot \arcsin(\frac{\sup A}{a}) + f(\sup A)^{\frac{1}{2}}) - (\inf A \cdot \arcsin(\frac{\inf A}{a}) + f(\inf A)^{\frac{1}{2}})$.

(70) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arccos) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arccos)$-(\square^{\frac{1}{2}}) \cdot f)$. Then $\int\limits_A$(the function arccos)$(x)dx = \sup A \cdot \arccos \sup A - f(\sup A)^{\frac{1}{2}} -$ $(\inf A \cdot \arccos \inf A - f(\inf A)^{\frac{1}{2}})$.

(71) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arccos) $\cdot f_3) = Z = \mathrm{dom}(\mathrm{id}_Z$ ((the function arccos) $\cdot f_3) - (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arccos) $\cdot f_3) \restriction A$ is continuous. Then $\int\limits_A$((the function arccos) $\cdot f_3)(x)dx =$ $\sup A \cdot \arccos(\frac{\sup A}{a}) - f(\sup A)^{\frac{1}{2}} - (\inf A \cdot \arccos(\frac{\inf A}{a}) - f(\inf A)^{\frac{1}{2}})$.

(72) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f_2 = \square^2$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) $Z = \operatorname{dom}(\text{the function arctan})$, and
(vi) $Z = \operatorname{dom}(\mathrm{id}_Z \text{ the function arctan} - \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))$.
Then $\int\limits_A (\text{the function arctan})(x)dx = \sup A \cdot \operatorname{arctan}\sup A - \frac{1}{2} \cdot \ln(1 + (\sup A)^2) - (\inf A \cdot \operatorname{arctan}\inf A - \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

(73) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) $f_2 = \square^2$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) $\operatorname{dom}(\text{the function arccot}) = Z$, and
(vi) $Z = \operatorname{dom}(\mathrm{id}_Z \text{ the function arccot} + \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))$.
Then $\int\limits_A (\text{the function arccot})(x)dx = (\sup A \cdot \operatorname{arccot}\sup A + \frac{1}{2} \cdot \ln(1 + (\sup A)^2)) - (\inf A \cdot \operatorname{arccot}\inf A + \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

References

[1] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[2] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[3] Chuanzhang Chen. *Mathematical Analysis*. Higher Education Press, Beijing, 1978.
[4] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.
[5] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.
[6] Artur Korniłowicz and Yasunari Shidama. Inverse trigonometric functions arcsin and arccos. *Formalized Mathematics*, 13(**1**):73–79, 2005.
[7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[8] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[9] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Xiquan Liang and Bing Xie. Inverse trigonometric functions arctan and arccot. *Formalized Mathematics*, 16(**2**):147–158, 2008, doi:10.2478/v10037-008-0021-3.
[12] Takashi Mitsuishi and Yuguang Yang. Properties of the trigonometric function. *Formalized Mathematics*, 8(**1**):103–106, 1999.
[13] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.
[14] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[15] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[16] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[18] Andrzej Trybulec and Yatsuka Nakamura. On the decomposition of a simple closed curve into two arcs. *Formalized Mathematics*, 10(**3**):163–167, 2002.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Peng Wang and Bo Li. Several differentiation formulas of special functions. Part V. *Formalized Mathematics*, 15(**3**):73–79, 2007, doi:10.2478/v10037-007-0009-4.
[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[23] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received October 14, 2008