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Several Integrability Formulas of Some Functions, Orthogonal Polynomials and Norm Functions

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Summary. In this article, we give several integrability formulas of some functions including the trigonometric function and the index function [3]. We also give the definitions of the orthogonal polynomial and norm function, and some of their important properties [19].

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The terminology and notation used here are introduced in the following articles: [10], [21], [17], [6], [20], [1], [9], [13], [2], [4], [18], [15], [5], [8], [11], [14], [12], [16], and [7].

For simplicity, we use the following convention: r, p, x denote real numbers, n denotes an element of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, f, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

We now state a number of propositions:

(1) $-(\text{the function exp}) \cdot ((-1)\square{+}0)$ is differentiable on $\mathbb{R}$ and for every x holds $(-(\text{the function exp}) \cdot ((-1)\square{+}0))'_{\restriction\mathbb{R}}(x) = \exp(-x)$.

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(2) $\int\limits_A ((\text{the function exp}) \cdot ((-1)\square{+}0))(x)dx = -\exp(-\sup A) + \exp(-\inf A)$.

(3) $\frac{1}{2}$ ((the function exp) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}$ ((the function exp) $\cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(2 \cdot x)$.

(4) $\int\limits_A ((\text{the function exp}) \cdot (2\square{+}0))(x)dx = \frac{1}{2} \cdot \exp(2 \cdot \sup A) - \frac{1}{2} \cdot \exp(2 \cdot \inf A)$.

(5) Suppose $r \neq 0$. Then $\frac{1}{r}$ ((the function exp) $\cdot(r\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{r}$ ((the function exp) $\cdot(r\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(r \cdot x)$.

(6) If $r \neq 0$, then $\int\limits_A ((\text{the function exp}) \cdot (r\square{+}0))(x)dx = \frac{1}{r} \cdot \exp(r \cdot \sup A) - \frac{1}{r} \cdot \exp(r \cdot \inf A)$.

(7) $\int\limits_A ((\text{the function sin}) \cdot (2\square{+}0))(x)dx = (-\frac{1}{2}) \cdot \cos(2 \cdot \sup A) - (-\frac{1}{2}) \cdot \cos(2 \cdot \inf A)$.

(8) Suppose $n \neq 0$. Then $(-\frac{1}{n})$ ((the function cos) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $((-\frac{1}{n})$ ((the function cos) $\cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \sin(n \cdot x)$.

(9) If $n \neq 0$, then $\int\limits_A ((\text{the function sin}) \cdot (n\square{+}0))(x)dx = (-\frac{1}{n}) \cdot \cos(n \cdot \sup A) - (-\frac{1}{n}) \cdot \cos(n \cdot \inf A)$.

(10) $\frac{1}{2}$ ((the function sin) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}$ ((the function sin) $\cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(2 \cdot x)$.

(11) $\int\limits_A ((\text{the function cos}) \cdot (2\square{+}0))(x)dx = \frac{1}{2} \cdot \sin(2 \cdot \sup A) - \frac{1}{2} \cdot \sin(2 \cdot \inf A)$.

(12) Suppose $n \neq 0$. Then $\frac{1}{n}$ ((the function sin) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n}$ ((the function sin) $\cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(n \cdot x)$.

(13) If $n \neq 0$, then $\int\limits_A ((\text{the function cos}) \cdot (n\square{+}0))(x)dx = \frac{1}{n} \cdot \sin(n \cdot \sup A) - \frac{1}{n} \cdot \sin(n \cdot \inf A)$.

(14) If $A \subseteq Z$, then $\int\limits_A (\mathrm{id}_Z \, (\text{the function sin}))(x)dx = ((-\sup A) \cdot \cos \sup A + \sin \sup A) - ((-\inf A) \cdot \cos \inf A + \sin \inf A)$.

(15) If $A \subseteq Z$, then $\int\limits_A (\mathrm{id}_Z \, (\text{the function cos}))(x)dx = (\sup A \cdot \sin \sup A + \cos \sup A) - (\inf A \cdot \sin \inf A + \cos \inf A)$.

(16) id_Z (the function cos) is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,(\text{the function cos}))'_{\restriction Z}(x) = \cos x - x \cdot \sin x$.

(17)(i) $-$the function sin $+\, \mathrm{id}_Z$ (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\text{the function sin} + \mathrm{id}_Z\,(\text{the function cos}))'_{\restriction Z}(x) = -x \cdot \sin x$.

(18) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)\,(\text{the function sin}))(x)dx = (-\sin \sup A + \sup A \cdot \cos \sup A) - (-\sin \inf A + \inf A \cdot \cos \inf A)$.

(19)(i) $-$the function cos $-\, \mathrm{id}_Z$ (the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-\text{the function cos} - \mathrm{id}_Z\,(\text{the function sin}))'_{\restriction Z}(x) = -x \cdot \cos x$.

(20) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)\,(\text{the function cos}))(x)dx = -\cos \sup A - \sup A \cdot \sin \sup A - (-\cos \inf A - \inf A \cdot \sin \inf A)$.

(21) If $A \subseteq Z$, then $\int\limits_A ((\text{the function sin}) + \mathrm{id}_Z\,(\text{the function cos}))(x)dx = \sup A \cdot \sin \sup A - \inf A \cdot \sin \inf A$.

(22) If $A \subseteq Z$, then $\int\limits_A (-\text{the function cos} + \mathrm{id}_Z\,(\text{the function sin}))(x)dx = (-\sup A) \cdot \cos \sup A - (-\inf A) \cdot \cos \inf A$.

(23) $\int\limits_A ((1\square{+}0)\,(\text{the function exp}))(x)dx = \exp(\sup A - 1) - \exp(\inf A - 1)$.

(24) $\frac{1}{n+1}\,(\square^{n+1})$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}\,(\square^{n+1}))'_{\restriction \mathbb{R}}(x) = x^n$.

(25) $\int\limits_A (\square^n)(x)dx = \dfrac{1}{n+1} \cdot (\sup A)^{n+1} - \dfrac{1}{n+1} \cdot (\inf A)^{n+1}$.

(26) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $(f - g) \restriction C = f \restriction C - g \restriction C$.

(27) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 + f_2) \restriction C)\,(g \restriction C) = (f_1\,g + f_2\,g) \restriction C$.

(28) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 - f_2) \restriction C)\,(g \restriction C) = (f_1\,g - f_2\,g) \restriction C$.

(29) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1\,f_2) \restriction C)\,(g \restriction C) = (f_1 \restriction C)\,((f_2\,g) \restriction C)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle f, g \rangle_A$ yielding a real number is defined by:

(Def. 1) $\langle f, g \rangle_A = \int\limits_A (f\,g)(x)dx$.

The following propositions are true:

(30) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f, g\rangle_A = \langle g, f\rangle_A$.

(31) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \upharpoonright A$ is total,
(ii) $(f_2\, g) \upharpoonright A$ is total,
(iii) $(f_1\, g) \upharpoonright A$ is bounded,
(iv) $f_1\, g$ is integrable on A,
(v) $(f_2\, g) \upharpoonright A$ is bounded, and
(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 + f_2, g\rangle_A = \langle (f_1), g\rangle_A + \langle (f_2), g\rangle_A$.

(32) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \upharpoonright A$ is total,
(ii) $(f_2\, g) \upharpoonright A$ is total,
(iii) $(f_1\, g) \upharpoonright A$ is bounded,
(iv) $f_1\, g$ is integrable on A,
(v) $(f_2\, g) \upharpoonright A$ is bounded, and
(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 - f_2, g\rangle_A = \langle (f_1), g\rangle_A - \langle (f_2), g\rangle_A$.

(33) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle -f, g\rangle_A = -\langle f, g\rangle_A$.

(34) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, g\rangle_A = r \cdot \langle f, g\rangle_A$.

(35) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\upharpoonright A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, p\, g\rangle_A = r \cdot p \cdot \langle f, g\rangle_A$.

(36) For all partial functions f, g, h from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f\, g, h\rangle_A = \langle f, g\, h\rangle_A$.

(37) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\, f) \upharpoonright A$ is total and $(f\, g) \upharpoonright A$ is total and $(g\, g) \upharpoonright A$ is total and $(f\, f) \upharpoonright A$ is bounded and $(f\, g) \upharpoonright A$ is bounded and $(g\, g) \upharpoonright A$ is bounded and $f\, f$ is integrable on A and $f\, g$ is integrable on A and $g\, g$ is integrable on A. Then $\langle f + g, f + g\rangle_A = \langle f, f\rangle_A + 2 \cdot \langle f, g\rangle_A + \langle g, g\rangle_A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that f is orthogonal with g in A if and only if:

(Def. 2) $\langle f, g\rangle_A = 0$.

The following propositions are true:

(38) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \upharpoonright A$ is total and $(f\,g) \upharpoonright A$ is total and $(g\,g) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $(f\,g) \upharpoonright A$ is bounded and $(g\,g) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A. Then $\langle f+g, f+g\rangle_A = \langle f, f\rangle_A + \langle g, g\rangle_A$.

(39) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$. Then $\langle f, f\rangle_A \geq 0$.

(40) The function sin is orthogonal with the function cos in $[0, \pi]$.

(41) The function sin is orthogonal with the function cos in $[0, \pi \cdot 2]$.

(42) The function sin is orthogonal with the function cos in $[2 \cdot n \cdot \pi, (2 \cdot n+1) \cdot \pi]$.

(43) The function sin is orthogonal with the function cos in $[x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$.

(44) The function sin is orthogonal with the function cos in $[-\pi, \pi]$.

(45) The function sin is orthogonal with the function cos in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(46) The function sin is orthogonal with the function cos in $[-2 \cdot \pi, 2 \cdot \pi]$.

(47) The function sin is orthogonal with the function cos in $[-2 \cdot n \cdot \pi, 2 \cdot n \cdot \pi]$.

(48) The function sin is orthogonal with the function cos in $[x - 2 \cdot n \cdot \pi, x + 2 \cdot n \cdot \pi]$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $||f||_A$ yields a real number and is defined by:

(Def. 3) $||f||_A = \sqrt{\langle f, f\rangle_A}$.

Next we state three propositions:

(49) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$. Then $0 \leq ||f||_A$.

(50) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $||1\,f||_A = ||f||_A$.

(51) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \upharpoonright A$ is total and $(f\,g) \upharpoonright A$ is total and $(g\,g) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $(f\,g) \upharpoonright A$ is bounded and $(g\,g) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$ and for every x such that $x \in A$ holds $((g\,g) \upharpoonright A)(x) \geq 0$. Then $(||f+g||_A)^{\mathbf{2}} = (||f||_A)^{\mathbf{2}} + (||g||_A)^{\mathbf{2}}$.

For simplicity, we follow the rules: a, b, x are real numbers, n is an element of $\mathbb{N}$, A is a closed-interval subset of $\mathbb{R}$, f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z is an open subset of $\mathbb{R}$.

Next we state several propositions:

(52) If $-a \notin A$, then $\frac{1}{1\square + a} \restriction A$ is continuous.

(53) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,
- (iii) $Z = \operatorname{dom} f$,
- (iv) $\operatorname{dom} f = \operatorname{dom} f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(a+x)^{\mathbf{2}}}$, and
- (vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(54) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,
- (iii) $\operatorname{dom}((-1)\,\frac{1}{f}) = Z$,
- (iv) $\operatorname{dom}((-1)\,\frac{1}{f}) = \operatorname{dom} f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a+x)^{\mathbf{2}}}$, and
- (vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -f(\sup A)^{-1} + f(\inf A)^{-1}$.

(55) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) \neq 0$,
- (iii) $\operatorname{dom} f = Z$,
- (iv) $\operatorname{dom} f = \operatorname{dom} f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a-x)^{\mathbf{2}}}$, and
- (vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(56) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
- (iii) $\operatorname{dom}((\text{the function } \ln) \cdot f) = Z$,
- (iv) $\operatorname{dom}((\text{the function } \ln) \cdot f) = \operatorname{dom} f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a+x}$, and
- (vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln(a + \sup A) - \ln(a + \inf A)$.

Next we state a number of propositions:

(57) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = x - a$ and $f(x) > 0$,

(iii) $\mathrm{dom}((\text{the function ln}) \cdot f) = Z$,

(iv) $\mathrm{dom}((\text{the function ln}) \cdot f) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x-a}$, and

(vi) $f_2{\restriction}A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln f(\sup A) - \ln f(\inf A)$.

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,

(iii) $\mathrm{dom}(-(\text{the function ln}) \cdot f) = Z$,

(iv) $\mathrm{dom}(-(\text{the function ln}) \cdot f) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a-x}$, and

(vi) $f_2{\restriction}A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln(a - \sup A) + \ln(a - \inf A)$.

(59) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - a\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x}{a+x}$ and $f_2{\restriction}A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - a \cdot f(\sup A) - (\inf A - a \cdot f(\inf A))$.

(60) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\mathrm{dom}((2 \cdot a)\, f - \mathrm{id}_Z) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{a-x}{a+x}$ and $f_2{\restriction}A$ is continuous. Then $\int\limits_A f_2(x)dx = 2 \cdot a \cdot f(\sup A) - \sup A - (2 \cdot a \cdot f(\inf A) - \inf A)$.

(61) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - (2 \cdot a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+a}$ and $f_2{\restriction}A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - 2 \cdot a \cdot f(\sup A) - (\inf A - 2 \cdot a \cdot f(\inf A))$.

(62) Suppose that $A \subseteq Z$ and $f = (\text{the function ln}) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (2 \cdot a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-a}$ and $f_2{\restriction}A$

is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + 2 \cdot a \cdot f(\sup A)) - (\inf A + 2 \cdot a \cdot f(\inf A))$.

(63) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a - b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (a - b) \cdot f(\sup A)) - (\inf A + (a - b) \cdot f(\inf A))$.

(64) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a + b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (a + b) \cdot f(\sup A)) - (\inf A + (a + b) \cdot f(\inf A))$.

(65) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - (a + b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - (a + b) \cdot f(\sup A) - (\inf A - (a + b) \cdot f(\inf A))$.

(66) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (b - a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (b - a) \cdot f(\sup A)) - (\inf A + (b - a) \cdot f(\inf A))$.

(67) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$,

(iii) dom((the function ln) $\cdot f) = Z$,

(iv) dom((the function ln) $\cdot f) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln \sup A - \ln \inf A$.

(68) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $x > 0$,

(iii) dom((the function ln) $\cdot(\square^n)) = Z$,

(iv) $\mathrm{dom}((\text{the function ln}) \cdot (\square^n)) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{n}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln((\sup A)^n) - \ln((\inf A)^n)$.

(69) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = x$,
(iii) $\mathrm{dom}((\text{the function ln}) \cdot \frac{1}{f}) = Z$,
(iv) $\mathrm{dom}((\text{the function ln}) \cdot \frac{1}{f}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln \sup A + \ln \inf A$.

(70) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) $\mathrm{dom}(\frac{2}{3}\, f^{\frac{3}{2}}) = Z$,
(iv) $\mathrm{dom}(\frac{2}{3}\, f^{\frac{3}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \frac{2}{3} \cdot (a + \sup A)^{\frac{3}{2}} - \frac{2}{3} \cdot (a + \inf A)^{\frac{3}{2}}$.

(71) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
(iii) $\mathrm{dom}((-\frac{2}{3})\, f^{\frac{3}{2}}) = Z$,
(iv) $\mathrm{dom}((-\frac{2}{3})\, f^{\frac{3}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\frac{2}{3} \cdot (a - \sup A)^{\frac{3}{2}} + \frac{2}{3} \cdot (a - \inf A)^{\frac{3}{2}}$.

(72) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) $\mathrm{dom}(2\, f^{\frac{1}{2}}) = Z$,
(iv) $\mathrm{dom}(2\, f^{\frac{1}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{-\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = 2\cdot(a+\sup A)^{\frac{1}{2}} - 2\cdot(a+\inf A)^{\frac{1}{2}}$.

(73) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
(iii) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = Z$,
(iv) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = \mathrm{dom}\, f_2$,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a-x)^{-\frac{1}{2}}$, and
(vi) $f_2{\restriction}A$ is continuous.

Then $\int\limits_A f_2(x)dx = -2\cdot(a-\sup A)^{\frac{1}{2}} + 2\cdot(a-\inf A)^{\frac{1}{2}}$.

(74) Suppose that
(i) $A \subseteq Z$,
(ii) $\mathrm{dom}((-\mathrm{id}_Z)\,(\text{the function cos}) + \text{the function sin}) = Z$,
(iii) for every x such that $x \in Z$ holds $f(x) = x \cdot \sin x$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (-\sup A \cdot \cos\sup A + \sin\sup A) - (-\inf A \cdot \cos\inf A + \sin\inf A)$.

(75) Suppose $A \subseteq Z$ and dom (the function sec) $= Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x}{(\cos x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = \sec\sup A - \sec\inf A$.

(76) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cosec})$. Then $-$the function cosec is differentiable on Z and for every x such that $x \in Z$ holds $(-\text{the function cosec})'_{\restriction Z}(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$.

(77) Suppose $A \subseteq Z$ and $\mathrm{dom}(-\text{the function cosec}) = Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = -\mathrm{cosec}\sup A + \mathrm{cosec}\inf A$.

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