FORMALIZED MATHEMATICS
Vol. **16**, No. **4**, Pages 371–376, **2008**
DOI: 10.2478/v10037-008-0045-8

BCI-homomorphisms

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Summary. In this article the notion of the power of an element of BCI-algebra and its period in the book [11], sections 1.4 to 1.5 are firstly given. Then the definition of BCI-homomorphism is defined and the fundamental theorem of homomorphism, the first isomorphism theorem and the second isomorphism theorem are proved following the book [9], section 1.6.

MML identifier: `BCIALG_6`, version: `7.9.03 4.108.1028`

The notation and terminology used in this paper have been introduced in the following articles: [6], [14], [3], [15], [5], [4], [2], [7], [10], [1], [13], [8], and [12].

1. The Power of an Element of BCI-algebras

In this paper X is a BCI-algebra and n is an element of $\mathbb{N}$.

Let D be a set, let f be a function from $\mathbb{N}$ into D, and let n be a natural number. Then $f(n)$ is an element of D.

Let G be a non empty BCI structure with 0. The functor BCI-power G yielding a function from (the carrier of G) $\times$ $\mathbb{N}$ into the carrier of G is defined as follows:

(Def. 1) For every element x of G holds (BCI-power G)$(x, 0) = 0_G$ and for every n holds (BCI-power G)$(x, n+1) = x \setminus$ (BCI-power G)$(x, n)^{\rm c}$.

ISSN 1426–2630(p), 1898-9934(e)

For simplicity, we adopt the following convention: x, y are elements of X, a, b are elements of AtomSet X, m, n are natural numbers, and i, j are integers.

Let us consider X, i, x. The functor x^i yielding an element of X is defined by:

(Def. 2) $x^i = \begin{cases} (\text{BCI-power}\,X)(x,\,|i|), & \text{if } 0 \leq i, \\ (\text{BCI-power}\,X)(x^{\mathrm{c}},\,|i|), & \text{otherwise.} \end{cases}$

Let us consider X, n, x. Then x^n can be characterized by the condition:

(Def. 3) $x^n = (\text{BCI-power}\,X)(x,\,n)$.

One can prove the following propositions:

(1) $a \setminus (x \setminus b) = b \setminus (x \setminus a)$.
(2) $x^{n+1} = x \setminus (x^n)^{\mathrm{c}}$.
(3) $x^0 = 0_X$.
(4) $x^1 = x$.
(5) $x^{-1} = x^{\mathrm{c}}$.
(6) $x^2 = x \setminus x^{\mathrm{c}}$.
(7) $(0_X)^n = 0_X$.
(8) $(a^{-1})^{-1} = a$.
(9) $x^{-n} = ((x^{\mathrm{c}})^{\mathrm{c}})^{-n}$.
(10) $(a^{\mathrm{c}})^n = a^{-n}$.
(11) If $x \in$ BCK-part X and $n \geq 1$, then $x^n = x$.
(12) If $x \in$ BCK-part X, then $x^{-n} = 0_X$.
(13) $a^i \in$ AtomSet X.
(14) $(a^{n+1})^{\mathrm{c}} = (a^n)^{\mathrm{c}} \setminus a$.
(15) $(a \setminus b)^n = a^n \setminus b^n$.
(16) $(a \setminus b)^{-n} = a^{-n} \setminus b^{-n}$.
(17) $(a^{\mathrm{c}})^n = (a^n)^{\mathrm{c}}$.
(18) $(x^{\mathrm{c}})^n = (x^n)^{\mathrm{c}}$.
(19) $(a^{\mathrm{c}})^{-n} = (a^{-n})^{\mathrm{c}}$.
(20) $x^n \in \mathrm{BranchV}(((x^{\mathrm{c}})^{\mathrm{c}})^n)$.
(21) $(x^n)^{\mathrm{c}} = (((x^{\mathrm{c}})^{\mathrm{c}})^n)^{\mathrm{c}}$.
(22) $a^i \setminus a^j = a^{i-j}$.
(23) $(a^i)^j = a^{i \cdot j}$.
(24) $a^{i+j} = a^i \setminus (a^j)^{\mathrm{c}}$.

Let us consider X, x. We say that x is finite-period if and only if:

(Def. 4) There exists an element n of $\mathbb{N}$ such that $n \neq 0$ and $x^n \in$ BCK-part X.

One can prove the following proposition

(25) If x is finite-period, then $(x^{\mathrm{c}})^{\mathrm{c}}$ is finite-period.

Let us consider X, x. Let us assume that x is finite-period. The functor $\operatorname{ord}(x)$ yielding an element of $\mathbb{N}$ is defined as follows:

(Def. 5) $x^{\operatorname{ord}(x)} \in$ BCK-part X and $\operatorname{ord}(x) \neq 0$ and for every element m of $\mathbb{N}$ such that $x^m \in$ BCK-part X and $m \neq 0$ holds $\operatorname{ord}(x) \leq m$.

One can prove the following propositions:

(26) If a is finite-period and $\operatorname{ord}(a) = n$, then $a^n = 0_X$.

(27) X is a BCK-algebra iff for every x holds x is finite-period and $\operatorname{ord}(x) = 1$.

(28) If x is finite-period and a is finite-period and $x \in$ BranchV a, then $\operatorname{ord}(x) = \operatorname{ord}(a)$.

(29) If x is finite-period and $\operatorname{ord}(x) = n$, then $x^m \in$ BCK-part X iff $n \mid m$.

(30) If x is finite-period and x^m is finite-period and $\operatorname{ord}(x) = n$ and $m > 0$, then $\operatorname{ord}(x^m) = n \div (m \operatorname{gcd} n)$.

(31) If x is finite-period and x^{c} is finite-period, then $\operatorname{ord}(x) = \operatorname{ord}(x^{\mathrm{c}})$.

(32) If $x \setminus y$ is finite-period and $x, y \in$ BranchV a, then $\operatorname{ord}(x \setminus y) = 1$.

(33) Suppose that $x \setminus y$ is finite-period and $a \setminus b$ is finite-period and x is finite-period and y is finite-period and a is finite-period and b is finite-period and $a \neq b$ and $x \in$ BranchV a and $y \in$ BranchV b. Then $\operatorname{ord}(a \setminus b) \mid \operatorname{lcm}(\operatorname{ord}(x), \operatorname{ord}(y))$.

2. Definition of BCI-homomorphisms

For simplicity, we follow the rules: X, X', Y, Z, W are BCI-algebras, H' denotes a subalgebra of X', G denotes a subalgebra of X, A' denotes a non empty subset of X', I denotes an ideal of X, C_1, K are closed ideals of X, x, y are elements of X, R_1 denotes an I-congruence of X by I, and R_2 denotes an I-congruence of X by K.

One can prove the following proposition

(34) Let X be a BCI-algebra, Y be a subalgebra of X, x, y be elements of X, and x', y' be elements of Y. If $x = x'$ and $y = y'$, then $x \setminus y = x' \setminus y'$.

Let X, X' be non empty BCI structures with 0 and let f be a function from X into X'. We say that f is multiplicative if and only if:

(Def. 6) For all elements a, b of X holds $f(a \setminus b) = f(a) \setminus f(b)$.

Let X, X' be BCI-algebras. Note that there exists a function from X into X' which is multiplicative.

Let X, X' be BCI-algebras. A BCI-homomorphism from X to X' is a multiplicative function from X into X'.

In the sequel f denotes a BCI-homomorphism from X to X', g denotes a BCI-homomorphism from X' to X, and h denotes a BCI-homomorphism from X' to Y.

Let us consider X, X', f. We say that f is isotonic if and only if:

(Def. 7) For all x, y such that $x \leq y$ holds $f(x) \leq f(y)$.

Let us consider X. An endomorphism of X is a BCI-homomorphism from X to X.

Let us consider X, X', f. The functor $\mathrm{Ker}\, f$ is defined by:

(Def. 8) $\mathrm{Ker}\, f = \{x \in X\colon f(x) = 0_{X'}\}$.

The following proposition is true

(35) $f(0_X) = 0_{X'}$.

Let us consider X, X', f. Observe that $\mathrm{Ker}\, f$ is non empty.

We now state several propositions:

(36) If $x \leq y$, then $f(x) \leq f(y)$.

(37) f is one-to-one iff $\mathrm{Ker}\, f = \{0_X\}$.

(38) If f is bijective and $g = f^{-1}$, then g is bijective.

(39) $h \cdot f$ is a BCI-homomorphism from X to Y.

(40) Let f be a BCI-homomorphism from X to Y, g be a BCI-homomorphism from Y to Z, and h be a BCI-homomorphism from Z to W. Then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(41) For every subalgebra Z of X' such that the carrier of $Z = \mathrm{rng}\, f$ holds f is a BCI-homomorphism from X to Z.

(42) $\mathrm{Ker}\, f$ is a closed ideal of X.

Let us consider X, X', f. Observe that $\mathrm{Ker}\, f$ is closed.

Next we state several propositions:

(43) If f is onto, then for every element c of X' there exists x such that $c = f(x)$.

(44) For every element a of X such that a is minimal holds $f(a)$ is minimal.

(45) For every element a of $\mathrm{AtomSet}\, X$ and for every element b of $\mathrm{AtomSet}\, X'$ such that $b = f(a)$ holds $f^\circ\, \mathrm{BranchV}\, a \subseteq \mathrm{BranchV}\, b$.

(46) If A' is an ideal of X', then $f^{-1}(A')$ is an ideal of X.

(47) If A' is a closed ideal of X', then $f^{-1}(A')$ is a closed ideal of X.

(48) If f is onto, then $f^\circ I$ is an ideal of X'.

(49) If f is onto, then $f^\circ C_1$ is a closed ideal of X'.

Let X, X' be BCI-algebras. We say that X and X' are isomorphic if and only if:

(Def. 9) There exists a BCI-homomorphism from X to X' which is bijective.

Let us consider X, let I be an ideal of X, and let R_1 be an I-congruence of X by I. Note that ${}^{X}/_{R_1}$ is strict, B, C, I, and BCI-4.

Let us consider X, let I be an ideal of X, and let R_1 be an I-congruence of X by I. The canonical homomorphism onto cosets of R_1 yielding a BCI-homomorphism from X to ${}^{X}/_{R_1}$ is defined as follows:

(Def. 10) For every x holds (the canonical homomorphism onto cosets of R_1)(x) $=$ $[x]_{(R_1)}$.

3. Fundamental Theorem of Homomorphisms

The following four propositions are true:

(50) The canonical homomorphism onto cosets of R_1 is onto.

(51) Suppose $I = \mathrm{Ker}\, f$. Then there exists a BCI-homomorphism h from ${}^{X}/_{R_1}$ to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.

(52) Let given X, X', I, R_1, f. Suppose $I = \mathrm{Ker}\, f$. Then there exists a BCI-homomorphism h from ${}^{X}/_{R_1}$ to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.

(53) Ker (the canonical homomorphism onto cosets of R_2) $= K$.

4. First Isomorphism Theorem

One can prove the following propositions:

(54) If $I = \mathrm{Ker}\, f$ and the carrier of $H' = \mathrm{rng}\, f$, then ${}^{X}/_{R_1}$ and H' are isomorphic.

(55) If $I = \mathrm{Ker}\, f$ and f is onto, then ${}^{X}/_{R_1}$ and X' are isomorphic.

5. Second Isomorphism Theorem

Let us consider X, G, K, R_2. The functor $\mathrm{Union}(G, R_2)$ yielding a non empty subset of X is defined by:

(Def. 11) $\mathrm{Union}(G, R_2) = \bigcup\{[a]_{(R_2)}; a$ ranges over elements of G: $[a]_{(R_2)} \in$ the carrier of ${}^{X}/_{R_2}\}$.

Let us consider X, G, K, R_2. The functor $\mathrm{HKOp}(G, R_2)$ yielding a binary operation on $\mathrm{Union}(G, R_2)$ is defined as follows:

(Def. 12) For all elements w_1, w_2 of $\mathrm{Union}(G, R_2)$ and for all elements x, y of X such that $w_1 = x$ and $w_2 = y$ holds $(\mathrm{HKOp}(G, R_2))(w_1, w_2) = x \setminus y$.

Let us consider X, G, K, R_2. The functor $\mathrm{zeroHK}(G, R_2)$ yields an element of $\mathrm{Union}(G, R_2)$ and is defined as follows:

(Def. 13) $\mathrm{zeroHK}(G, R_2) = 0_X$.

Let us consider X, G, K, R_2. The functor $\mathrm{HK}(G, R_2)$ yielding a BCI structure with 0 is defined as follows:

(Def. 14) $\mathrm{HK}(G, R_2) = \langle \mathrm{Union}(G, R_2), \mathrm{HKOp}(G, R_2), \mathrm{zeroHK}(G, R_2) \rangle$.

Let us consider X, G, K, R_2. Observe that $\mathrm{HK}(G, R_2)$ is non empty.

Let us consider X, G, K, R_2 and let w_1, w_2 be elements of $\mathrm{Union}(G, R_2)$. The functor $w_1 \setminus w_2$ yielding an element of $\mathrm{Union}(G, R_2)$ is defined by:

(Def. 15) $w_1 \setminus w_2 = (\mathrm{HKOp}(G, R_2))(w_1, w_2)$.

We now state the proposition

(56) $\mathrm{HK}(G, R_2)$ is a BCI-algebra.

Let us consider X, G, K, R_2. Observe that $\mathrm{HK}(G, R_2)$ is strict, B, C, I, and BCI-4.

We now state three propositions:

(57) $\mathrm{HK}(G, R_2)$ is a subalgebra of X.

(58) (The carrier of G) $\cap K$ is a closed ideal of G.

(59) Let K_1 be an ideal of $\mathrm{HK}(G, R_2)$, R_3 be an I-congruence of $\mathrm{HK}(G, R_2)$ by K_1, I be an ideal of G, and R_1 be an I-congruence of G by I. Suppose $K_1 = K$ and $R_3 = R_2$ and $I =$ (the carrier of G) $\cap K$. Then ${}^{G}/_{R_1}$ and ${}^{\mathrm{HK}(G,R_2)}/_{R_3}$ are isomorphic.

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Received August 26, 2008