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On L^1 Space Formed by Real-Valued Partial Functions

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Summary. This article contains some definitions and properties refering to function spaces formed by partial functions defined over a measurable space. We formalized a function space, the so-called L^1 space and proved that the space turns out to be a normed space. The formalization of a real function space was given in [16]. The set of all function forms additive group. Here addition is defined by point-wise addition of two functions. However it is not true for partial functions. The set of partial functions does not form an additive group due to lack of right zeroed condition. Therefore, firstly we introduced a kind of a quasi-linear space, then, we introduced the definition of an equivalent relation of two functions which are almost everywhere equal ($=_{\text{a.e.}}$), thirdly we formalized a linear space by taking the quotient of a quasi-linear space by the relation ($=_{\text{a.e.}}$).

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The papers [11], [24], [4], [5], [3], [8], [25], [10], [9], [14], [7], [20], [13], [23], [22], [1], [17], [21], [18], [15], [6], [12], [19], and [2] provide the notation and terminology for this paper.

1. Preliminaries of Real Linear Space

Let V be a non empty RLS structure and let V_1 be a subset of V. We say that V_1 is multiplicatively-closed if and only if:

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(Def. 1) For every real number a and for every vector v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

The following proposition is true

(1) Let V be a real linear space and V_1 be a subset of V. Then V_1 is linearly closed if and only if V_1 is add closed and multiplicatively-closed.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is add closed, multiplicatively-closed, and non empty.

Let X be a non empty RLS structure and let X_1 be a multiplicatively-closed non empty subset of X. The functor $\cdot_{(X_1)}$ yields a function from $\mathbb{R} \times X_1$ into X_1 and is defined by:

(Def. 2) $\cdot_{(X_1)} = (\text{the external multiplication of } X)\restriction(\mathbb{R} \times X_1)$.

In the sequel a, b, r denote real numbers.

Next we state four propositions:

(2) Let V be an Abelian add-associative right zeroed real linear space-like non empty RLS structure, V_1 be a non empty subset of V, d_1 be an element of V_1, A be a binary operation on V_1, and M be a function from $\mathbb{R} \times V_1$ into V_1. Suppose $d_1 = 0_V$ and $A = (\text{the addition of } V) \restriction (V_1)$ and $M = (\text{the external multiplication of } V)\restriction(\mathbb{R} \times V_1)$. Then $\langle V_1, d_1, A, M\rangle$ is Abelian, add-associative, right zeroed, and real linear space-like.

(3) Let V be an Abelian add-associative right zeroed real linear space-like non empty RLS structure and V_1 be an add closed multiplicatively-closed non empty subset of V. Suppose $0_V \in V_1$. Then $\langle V_1, 0_V(\in V_1), \text{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$ is Abelian, add-associative, right zeroed, and real linear space-like.

(4) Let V be a non empty RLS structure, V_1 be an add closed multiplicatively-closed non empty subset of V, v, u be vectors of V, and w_1, w_2 be vectors of $\langle V_1, 0_V(\in V_1), \text{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(5) Let V be a non empty RLS structure, V_1 be an add closed multiplicatively-closed non empty subset of V, a be a real number, v be a vector of V, and w be a vector of $\langle V_1, 0_V(\in V_1), \text{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$. If $w = v$, then $a \cdot w = a \cdot v$.

2. Quasi-Real Linear Space of Partial Functions

We adopt the following convention: A, B denote non empty sets and f, g, h denote elements of $A \dot{\to} \mathbb{R}$.

Let us consider A, B, let F be a binary operation on $A \dot{\to} B$, and let f, g be elements of $A \dot{\to} B$. Then $F(f, g)$ is an element of $A \dot{\to} B$.

Let us consider A. The functor $\cdot_{A \dot{\to} \mathbb{R}}$ yielding a binary operation on $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 3) For all elements f, g of $A \dot{\to} \mathbb{R}$ holds $\cdot_{A \dot{\to} \mathbb{R}}(f, g) = f\, g$.

Let us consider A. The functor $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}$ yielding a function from $\mathbb{R} \times (A \dot{\to} \mathbb{R})$ into $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 4) For every real number a and for every element f of $A \dot{\to} \mathbb{R}$ holds $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f) = a\, f$.

Let us consider A. The functor $0_{A \dot{\to} \mathbb{R}}$ yielding an element of $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 5) $0_{A \dot{\to} \mathbb{R}} = A \longmapsto 0$.

Let us consider A. The functor $1_{A \dot{\to} \mathbb{R}}$ yields an element of $A \dot{\to} \mathbb{R}$ and is defined as follows:

(Def. 6) $1_{A \dot{\to} \mathbb{R}} = A \longmapsto 1$.

The following propositions are true:

(6) $h = +_{A \dot{\to} \mathbb{R}}(f, g)$ iff $\operatorname{dom} h = \operatorname{dom} f \cap \operatorname{dom} g$ and for every element x of A such that $x \in \operatorname{dom} h$ holds $h(x) = f(x) + g(x)$.

(7) $h = \cdot_{A \dot{\to} \mathbb{R}}(f, g)$ iff $\operatorname{dom} h = \operatorname{dom} f \cap \operatorname{dom} g$ and for every element x of A such that $x \in \operatorname{dom} h$ holds $h(x) = f(x) \cdot g(x)$.

(8) $0_{A \dot{\to} \mathbb{R}} \neq 1_{A \dot{\to} \mathbb{R}}$.

(9) $h = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f)$ iff $\operatorname{dom} h = \operatorname{dom} f$ and for every element x of A such that $x \in \operatorname{dom} f$ holds $h(x) = a \cdot f(x)$.

(10) $+_{A \dot{\to} \mathbb{R}}(f, g) = +_{A \dot{\to} \mathbb{R}}(g, f)$.

(11) $+_{A \dot{\to} \mathbb{R}}(f, +_{A \dot{\to} \mathbb{R}}(g, h)) = +_{A \dot{\to} \mathbb{R}}(+_{A \dot{\to} \mathbb{R}}(f, g), h)$.

(12) $\cdot_{A \dot{\to} \mathbb{R}}(f, g) = \cdot_{A \dot{\to} \mathbb{R}}(g, f)$.

(13) $\cdot_{A \dot{\to} \mathbb{R}}(f, \cdot_{A \dot{\to} \mathbb{R}}(g, h)) = \cdot_{A \dot{\to} \mathbb{R}}(\cdot_{A \dot{\to} \mathbb{R}}(f, g), h)$.

(14) $\cdot_{A \dot{\to} \mathbb{R}}(1_{A \dot{\to} \mathbb{R}}, f) = f$.

(15) $+_{A \dot{\to} \mathbb{R}}(0_{A \dot{\to} \mathbb{R}}, f) = f$.

(16) $+_{A \dot{\to} \mathbb{R}}(f, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(-1, f)) = 0_{A \dot{\to} \mathbb{R}} \restriction \operatorname{dom} f$.

(17) $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(1, f) = f$.

(18) $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(b, f)) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a \cdot b, f)$.

(19) $+_{A \dot{\to} \mathbb{R}}(\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f), \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(b, f)) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a + b, f)$.

(20) $\cdot_{A \dot{\to} \mathbb{R}}(f, +_{A \dot{\to} \mathbb{R}}(g, h)) = +_{A \dot{\to} \mathbb{R}}(\cdot_{A \dot{\to} \mathbb{R}}(f, g), \cdot_{A \dot{\to} \mathbb{R}}(f, h))$.

(21) $\cdot_{A \dot{\to} \mathbb{R}}(\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f), g) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, \cdot_{A \dot{\to} \mathbb{R}}(f, g))$.

Let us consider A. The functor $\mathrm{PFunct}_{\mathrm{RLS}}\, A$ yields a non empty RLS structure and is defined by:

(Def. 7) $\mathrm{PFunct}_{\mathrm{RLS}}\, A = \langle A \dot{\to} \mathbb{R}, 0_{A \dot{\to} \mathbb{R}}, +_{A \dot{\to} \mathbb{R}}, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}} \rangle$.

Let us consider A. One can verify that $\mathrm{PFunct}_{\mathrm{RLS}}\, A$ is strict, Abelian, add-associative, right zeroed, and real linear space-like.

3. Quasi-Real Linear Space of Integrable Functions

For simplicity, we use the following convention: X is a non empty set, x is an element of X, S is a σ-field of subsets of X, M is a σ-measure on S, E is an element of S, and f, g, h, f_1, g_1 are partial functions from X to $\mathbb{R}$.

Next we state the proposition

(22) Let given X, S, M and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists E such that $E = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $0 = f(x)$. Then f is integrable on M and $\int f \,\mathrm{d}M = 0$.

Let X be a non empty set and let r be a real number. Then $X \longmapsto r$ is a partial function from X to $\mathbb{R}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The L^1 functions of M yielding a non empty subset of $\mathrm{PFunct}_{\mathrm{RLS}}\, X$ is defined by the condition (Def. 8).

(Def. 8) The L^1 functions of $M = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $\bigvee_{N_1 : \text{element of } S} (M(N_1) = 0 \wedge \mathrm{dom}\, f = {N_1}^{\mathrm{c}} \wedge f$ is integrable on $M)\}$.

We now state two propositions:

(23) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M. Then $f + g \in$ the L^1 functions of M.

(24) If $f \in$ the L^1 functions of M, then $a\, f \in$ the L^1 functions of M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Observe that the L^1 functions of M is multiplicatively-closed and add closed.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L^1-$\mathrm{Funct}_{\mathrm{RLS}} M$ yielding a non empty RLS structure is defined by the condition (Def. 9).

(Def. 9) L^1-$\mathrm{Funct}_{\mathrm{RLS}} M = \langle$the L^1 functions of M, $0_{\mathrm{PFunct}_{\mathrm{RLS}}\, X}(\in$ the L^1 functions of $M)$, add $|$(the L^1 functions of M, $\mathrm{PFunct}_{\mathrm{RLS}}\, X)$, $\cdot_{\text{the } L^1 \text{ functions of } M}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Observe that L^1-$\mathrm{Funct}_{\mathrm{RLS}} M$ is strict, Abelian, add-associative, right zeroed, and real linear space-like.

4. Quotient Space of Quasi-Real Linear Space of Integrable Functions

In the sequel v, u are vectors of L^1-$\mathrm{Funct}_{\mathrm{RLS}} M$.

Next we state two propositions:

(25) $(v) + (u) = v + u$.

(26) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f, g be partial functions from X to $\mathbb{R}$. The predicate $f =^{M}_{\text{a.e.}} g$ is defined by:

(Def. 10) There exists an element E of S such that $M(E) = 0$ and $f\restriction E^{\mathrm{c}} = g\restriction E^{\mathrm{c}}$.

We now state several propositions:

(27) Suppose $f = u$. Then

(i) $u + (-1) \cdot u = (X \longmapsto 0)\restriction \operatorname{dom} f$, and

(ii) there exist partial functions v, g from X to $\mathbb{R}$ such that $v \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $v = u + (-1) \cdot u$ and $g = X \longmapsto 0$ and $v =^{M}_{\text{a.e.}} g$.

(28) $f =^{M}_{\text{a.e.}} f$.

(29) If $f =^{M}_{\text{a.e.}} g$, then $g =^{M}_{\text{a.e.}} f$.

(30) If $f =^{M}_{\text{a.e.}} g$ and $g =^{M}_{\text{a.e.}} h$, then $f =^{M}_{\text{a.e.}} h$.

(31) If $f =^{M}_{\text{a.e.}} f_1$ and $g =^{M}_{\text{a.e.}} g_1$, then $f + g =^{M}_{\text{a.e.}} f_1 + g_1$.

(32) If $f =^{M}_{\text{a.e.}} g$, then $a\, f =^{M}_{\text{a.e.}} a\, g$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor AlmostZeroFunctions M yielding a non empty subset of L^1-$\mathrm{Funct}_{\mathrm{RLS}} M$ is defined as follows:

(Def. 11) AlmostZeroFunctions $M = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in$ the L^1 functions of $M\ \wedge\ f =^{M}_{\text{a.e.}} X \longmapsto 0\}$.

The following proposition is true

(33) $(X \longmapsto 0) + (X \longmapsto 0) = X \longmapsto 0$ and $a\,(X \longmapsto 0) = X \longmapsto 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that AlmostZeroFunctions M is add closed and multiplicatively-closed.

Next we state the proposition

(34) $0_{L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M} = X \longmapsto 0$ and $0_{L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M} \in$ AlmostZeroFunctions M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{AlmostZeroFunct}_{\mathrm{RLS}}\, M$ yielding a non empty RLS structure is defined as follows:

(Def. 12) $\mathrm{AlmostZeroFunct}_{\mathrm{RLS}}\, M = \langle \text{AlmostZeroFunctions}\, M, 0_{L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M}(\in \text{AlmostZeroFunctions}\, M), \text{add}\,|(\text{AlmostZeroFunctions}\, M, L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M), \cdot_{\text{AlmostZeroFunctions}\, M}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Note that L^1-$\mathrm{Funct}_{\mathrm{RLS}} M$ is strict, strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel v, u are vectors of $\mathrm{AlmostZeroFunct}_{\mathrm{RLS}}\, M$.

Next we state two propositions:

(35) $(v) + (u) = v + u$.

(36) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. The functor $[f]^{M}_{\text{a.e.}}$ yielding a subset of the L^1 functions of M is defined by the condition (Def. 13).

(Def. 13) $[f]^{M}_{\text{a.e.}} = \{g; g$ ranges over partial functions from X to $\mathbb{R}$: $g \in$ the L^1 functions of $M \wedge f \in$ the L^1 functions of $M \wedge f =^{M}_{\text{a.e.}} g\}$.

The following propositions are true:

(37) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M, then $g =^{M}_{\text{a.e.}} f$ iff $g \in [f]^{M}_{\text{a.e.}}$.

(38) If $f \in$ the L^1 functions of M, then $f \in [f]^{M}_{\text{a.e.}}$.

(39) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M, then $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$ iff $f =^{M}_{\text{a.e.}} g$.

(40) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M. Then $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$ if and only if $g \in [f]^{M}_{\text{a.e.}}$.

(41) Suppose that

(i) $f \in$ the L^1 functions of M,

(ii) $f_1 \in$ the L^1 functions of M,

(iii) $g \in$ the L^1 functions of M,

(iv) $g_1 \in$ the L^1 functions of M,

(v) $[f]^{M}_{\text{a.e.}} = [f_1]^{M}_{\text{a.e.}}$, and

(vi) $[g]^{M}_{\text{a.e.}} = [g_1]^{M}_{\text{a.e.}}$.

Then $[f + g]^{M}_{\text{a.e.}} = [f_1 + g_1]^{M}_{\text{a.e.}}$.

(42) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$, then $[a\,f]^{M}_{\text{a.e.}} = [a\,g]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor CosetSet M yields a non empty family of subsets of the L^1 functions of M and is defined by:

(Def. 14) CosetSet $M = \{[f]^{M}_{\text{a.e.}}; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in$ the L^1 functions of $M\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor addCoset M yields a binary operation on CosetSet M and is defined by the condition (Def. 15).

(Def. 15) Let A, B be elements of CosetSet M and a, b be partial functions from X to $\mathbb{R}$. If $a \in A$ and $b \in B$, then $(\text{addCoset}\,M)(A, B) = [a + b]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor zeroCoset M yielding an element of CosetSet M is defined by:

(Def. 16) There exists a partial function f from X to $\mathbb{R}$ such that $f = X \longmapsto 0$ and $f \in$ the L^1 functions of M and zeroCoset $M = [f]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor lmultCoset M yields a function from $\mathbb{R} \times$ CosetSet M into CosetSet M and is defined by the condition (Def. 17).

(Def. 17) Let z be an element of $\mathbb{R}$, A be an element of CosetSet M, and f be a partial function from X to $\mathbb{R}$. If $f \in A$, then $(\text{lmultCoset}\, M)(z, A) = [z\, f]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor pre-L-Space M yields a strict Abelian add-associative right zeroed right complementable real linear space-like non empty RLS structure and is defined by the conditions (Def. 18).

(Def. 18)(i) The carrier of pre-L-Space M = CosetSet M,

(ii) the addition of pre-L-Space M = addCoset M,

(iii) $0_{\text{pre-}L\text{-Space}\, M}$ = zeroCoset M, and

(iv) the external multiplication of pre-L-Space M = lmultCoset M.

5. Real Normed Space of Integrable Functions

One can prove the following propositions:

(43) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $f =^{M}_{\text{a.e.}} g$, then $\int f\, \mathrm{d}M = \int g\, \mathrm{d}M$.

(44) If f is integrable on M, then $\int f\, \mathrm{d}M$, $\int |f|\, \mathrm{d}M \in \mathbb{R}$ and $|f|$ is integrable on M.

(45) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $f =^{M}_{\text{a.e.}} g$. Then $|f| =^{M}_{\text{a.e.}} |g|$ and $\int |f|\, \mathrm{d}M = \int |g|\, \mathrm{d}M$.

(46) Given a vector x of pre-L-Space M such that $f, g \in x$. Then $f =^{M}_{\text{a.e.}} g$ and $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M.

(47) There exists a function N_2 from the carrier of pre-L-Space M into $\mathbb{R}$ such that for every point x of pre-L-Space M holds there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and $N_2(x) = \int |f|\, \mathrm{d}M$.

In the sequel x is a point of pre-L-Space M.

The following two propositions are true:

(48) If $f \in x$, then f is integrable on M and $f \in$ the L^1 functions of M and $|f|$ is integrable on M.

(49) If $f, g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $\int f\, \mathrm{d}M = \int g\, \mathrm{d}M$ and $\int |f|\, \mathrm{d}M = \int |g|\, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L^1-Norm(M) yields a function from the carrier of pre-L-Space M into $\mathbb{R}$ and is defined by:

(Def. 19) For every point x of pre-L-Space M there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and $(L^1\text{-Norm}(M))(x) = \int |f|\, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L^1-Space(M) yielding a non empty strict normed structure is defined by:

(Def. 20) The RLS structure of L^1-Space(M) = pre-L-Space M and the norm of L^1-Space$(M) = L^1$-Norm(N).

In the sequel x, y are points of L^1-Space(M).

Next we state several propositions:

(50)(i) There exists a partial function f from X to $\mathbb{R}$ such that $f \in$ the L^1 functions of M and $x = [f]^M_{\text{a.e.}}$ and $\|x\| = \int |f| \, \mathrm{d}M$, and

(ii) for every partial function f from X to $\mathbb{R}$ such that $f \in x$ holds $\int |f| \, \mathrm{d}M = \|x\|$.

(51) If $f \in x$, then $x = [f]^M_{\text{a.e.}}$ and $\|x\| = \int |f| \, \mathrm{d}M$.

(52) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a\, f \in a \cdot x$.

(53) If $E = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$, then f is measurable on E.

(54) If $f \in$ the L^1 functions of M and $\int |f| \, \mathrm{d}M = 0$, then $f =^M_{\text{a.e.}} X \longmapsto 0$.

(55) $\int |X \longmapsto 0| \, \mathrm{d}M = 0$.

(56) If f is integrable on M and g is integrable on M, then $\int |f + g| \, \mathrm{d}M \leq \int |f| \, \mathrm{d}M + \int |g| \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that L^1-Space(M) is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

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