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Basic Properties of Circulant Matrices and Anti-Circular Matrices

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Summary. This article introduces definitions of circulant matrices, line- and column-circulant matrices as well as anti-circular matrices and describes their main properties.

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The articles [6], [9], [4], [10], [1], [14], [13], [2], [5], [8], [12], [11], [3], and [7] provide the notation and terminology for this paper.

1. Some Properties of Circulant Matrices

For simplicity, we adopt the following convention: i, j, k, n, l denote elements of $\mathbb{N}$, K denotes a field, a, b, c denote elements of K, p, q denote finite sequences of elements of K, and M_1, M_2, M_3 denote square matrices over K of dimension n.

Next we state two propositions:

(1) $\mathbf{1}_K \cdot p = p$.

(2) $(-\mathbf{1}_K) \cdot p = -p$.

Let K be a set, let M be a matrix over K, and let p be a finite sequence. We say that M is line circulant about p if and only if:

(Def. 1) $\operatorname{len} p = \operatorname{width} M$ and for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = p(((j - i) \bmod \operatorname{len} p) + 1)$.

Let K be a set and let M be a matrix over K. We say that M is line circulant if and only if:

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(Def. 2) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{width} M$ and M is line circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. We say that p is first-line-of-circulant if and only if:

(Def. 3) There exists a square matrix over K of dimension $\operatorname{len} p$ which is line circulant about p.

Let K be a set, let M be a matrix over K, and let p be a finite sequence. We say that M is column circulant about p if and only if:

(Def. 4) $\operatorname{len} p = \operatorname{len} M$ and for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = p(((i-j) \bmod \operatorname{len} p)+1)$.

Let K be a set and let M be a matrix over K. We say that M is column circulant if and only if:

(Def. 5) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{len} M$ and M is column circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. We say that p is first-column-of-circulant if and only if:

(Def. 6) There exists a square matrix over K of dimension $\operatorname{len} p$ which is column circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. Let us assume that p is first-line-of-circulant. The functor $\operatorname{LCirc} p$ yields a square matrix over K of dimension $\operatorname{len} p$ and is defined by:

(Def. 7) $\operatorname{LCirc} p$ is line circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. Let us assume that p is first-column-of-circulant. The functor $\operatorname{CCirc} p$ yielding a square matrix over K of dimension $\operatorname{len} p$ is defined by:

(Def. 8) $\operatorname{CCirc} p$ is column circulant about p.

Let K be a field. One can verify that there exists a finite sequence of elements of K which is first-line-of-circulant and first-column-of-circulant.

Let us consider K, n. Observe that $0_K^{n\times n}$ is line circulant and column circulant.

Let us consider K, let us consider n, and let a be an element of K. Observe that $(a)^{n\times n}$ is line circulant and $(a)^{n\times n}$ is column circulant.

Let us consider K. Note that there exists a matrix over K which is line circulant and column circulant.

In the sequel D denotes a non empty set, t denotes a finite sequence of elements of D, and A denotes a square matrix over D of dimension n.

We now state a number of propositions:

(3) If A is line circulant and $n > 0$, then A^{T} is column circulant.

(4) If A is line circulant about t and $n > 0$, then $t = \operatorname{Line}(A, 1)$.

(5) If A is line circulant and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i+1$ and $l = j+1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.

(6) If M_1 is line circulant, then $a \cdot M_1$ is line circulant.

(7) If M_1 is line circulant and M_2 is line circulant, then $M_1 + M_2$ is line circulant.

(8) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $M_1 + M_2 + M_3$ is line circulant.

(9) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 + b \cdot M_2$ is line circulant.

(10) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is line circulant.

(11) If M_1 is line circulant, then $-M_1$ is line circulant.

(12) If M_1 is line circulant and M_2 is line circulant, then $M_1 - M_2$ is line circulant.

(13) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 - b \cdot M_2$ is line circulant.

(14) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is line circulant.

(15) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is line circulant.

(16) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is line circulant.

(17) If A is column circulant and $n > 0$, then A^{T} is line circulant.

(18) If A is column circulant about t and $n > 0$, then $t = A_{\square,1}$.

(19) If A is column circulant and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i + 1$ and $l = j + 1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.

(20) If M_1 is column circulant, then $a \cdot M_1$ is column circulant.

(21) If M_1 is column circulant and M_2 is column circulant, then $M_1 + M_2$ is column circulant.

(22) If M_1 is column circulant and M_2 is column circulant and M_3 is column circulant, then $M_1 + M_2 + M_3$ is column circulant.

(23) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 + b \cdot M_2$ is column circulant.

(24) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is column circulant.

(25) If M_1 is column circulant, then $-M_1$ is column circulant.

(26) If M_1 is column circulant and M_2 is column circulant, then $M_1 - M_2$ is column circulant.

(27) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 - b \cdot M_2$ is column circulant.

(28) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is column circulant.

(29) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is column circulant.

(30) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is column circulant.

(31) If p is first-line-of-circulant, then $-p$ is first-line-of-circulant.

(32) If p is first-line-of-circulant, then $\mathrm{LCirc}(-p) = -\mathrm{LCirc}\, p$.

(33) Suppose p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-line-of-circulant.

(34) If $\mathrm{len}\, p = \mathrm{len}\, q$ and p is first-line-of-circulant and q is first-line-of-circulant, then $\mathrm{LCirc}(p + q) = \mathrm{LCirc}\, p + \mathrm{LCirc}\, q$.

(35) If p is first-column-of-circulant, then $-p$ is first-column-of-circulant.

(36) For every finite sequence p of elements of K such that p is first-column-of-circulant holds $\mathrm{CCirc}(-p) = -\mathrm{CCirc}\, p$.

(37) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-column-of-circulant.

(38) If $\mathrm{len}\, p = \mathrm{len}\, q$ and p is first-column-of-circulant and q is first-column-of-circulant, then $\mathrm{CCirc}(p + q) = \mathrm{CCirc}\, p + \mathrm{CCirc}\, q$.

(39) If $n > 0$, then $I_K^{n \times n}$ is column circulant.

(40) If $n > 0$, then $I_K^{n \times n}$ is line circulant.

(41) If p is first-line-of-circulant, then $a \cdot p$ is first-line-of-circulant.

(42) If p is first-line-of-circulant, then $\mathrm{LCirc}(a \cdot p) = a \cdot \mathrm{LCirc}\, p$.

(43) If p is first-line-of-circulant, then $a \cdot \mathrm{LCirc}\, p + b \cdot \mathrm{LCirc}\, p = \mathrm{LCirc}((a+b) \cdot p)$.

(44) If p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$, then $a \cdot \mathrm{LCirc}\, p + a \cdot \mathrm{LCirc}\, q = \mathrm{LCirc}(a \cdot (p + q))$.

(45) If p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$, then $a \cdot \mathrm{LCirc}\, p + b \cdot \mathrm{LCirc}\, q = \mathrm{LCirc}(a \cdot p + b \cdot q)$.

(46) If p is first-column-of-circulant, then $a \cdot p$ is first-column-of-circulant.

(47) If p is first-column-of-circulant, then $\mathrm{CCirc}(a \cdot p) = a \cdot \mathrm{CCirc}\, p$.

(48) If p is first-column-of-circulant, then $a \cdot \mathrm{CCirc}\, p + b \cdot \mathrm{CCirc}\, p = \mathrm{CCirc}((a + b) \cdot p)$.

(49) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$. Then $a \cdot \mathrm{CCirc}\, p + a \cdot \mathrm{CCirc}\, q = \mathrm{CCirc}(a \cdot (p + q))$.

(50) If p is first-column-of-circulant and q is first-column-of-circulant and $\operatorname{len} p = \operatorname{len} q$, then $a \cdot \operatorname{CCirc} p + b \cdot \operatorname{CCirc} q = \operatorname{CCirc}(a \cdot p + b \cdot q)$.

Let K be a set and let M be a matrix over K. We introduce M is circulant as a synonym of M is line circulant.

2. Some Properties of Anti-circular Matrices

Let K be a field, let M_1 be a matrix over K, and let p be a finite sequence of elements of K. We say that M_1 is anti-circular about p if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $\operatorname{len} p = \operatorname{width} M_1$,

(ii) for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M_1 and $i \leq j$ holds $(M_1)_{i,j} = p(((j - i) \bmod \operatorname{len} p) + 1)$, and

(iii) for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M_1 and $i \geq j$ holds $(M_1)_{i,j} = (-p)(((j - i) \bmod \operatorname{len} p) + 1)$.

Let K be a field and let M be a matrix over K. We say that M is anti-circular if and only if:

(Def. 10) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{width} M$ and M is anti-circular about p.

Let K be a field and let p be a finite sequence of elements of K. We say that p is first-line-of-anti-circular if and only if:

(Def. 11) There exists a square matrix over K of dimension $\operatorname{len} p$ which is anti-circular about p.

Let K be a field and let p be a finite sequence of elements of K. Let us assume that p is first-line-of-anti-circular. The functor $\operatorname{ACirc} p$ yields a square matrix over K of dimension $\operatorname{len} p$ and is defined by:

(Def. 12) $\operatorname{ACirc} p$ is anti-circular about p.

One can prove the following propositions:

(51) If M_1 is anti-circular, then $a \cdot M_1$ is anti-circular.

(52) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 + M_2$ is anti-circular.

(53) Let K be a Fanoian field, n, i, j be natural numbers, and M_1 be a square matrix over K of dimension n. Suppose $\langle i, j\rangle \in$ the indices of M_1 and $i = j$ and M_1 is anti-circular. Then $(M_1)_{i,j} = 0_K$.

(54) If M_1 is anti-circular and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i+1$ and $l = j+1$ and $i < n$ and $j < n$, then $(M_1)_{k,l} = (M_1)_{i,j}$.

(55) If M_1 is anti-circular, then $-M_1$ is anti-circular.

(56) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 - M_2$ is anti-circular.

(57) If M_1 is anti-circular about p and $n > 0$, then $p = \mathrm{Line}(M_1, 1)$.

(58) If p is first-line-of-anti-circular, then $-p$ is first-line-of-anti-circular.

(59) If p is first-line-of-anti-circular, then $\mathrm{ACirc}(-p) = -\mathrm{ACirc}\, p$.

(60) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-line-of-anti-circular.

(61) If p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$, then $\mathrm{ACirc}(p + q) = \mathrm{ACirc}\, p + \mathrm{ACirc}\, q$.

(62) If p is first-line-of-anti-circular, then $a \cdot p$ is first-line-of-anti-circular.

(63) If p is first-line-of-anti-circular, then $\mathrm{ACirc}(a \cdot p) = a \cdot \mathrm{ACirc}\, p$.

(64) If p is first-line-of-anti-circular, then $a \cdot \mathrm{ACirc}\, p + b \cdot \mathrm{ACirc}\, p = \mathrm{ACirc}((a + b) \cdot p)$.

(65) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$. Then $a \cdot \mathrm{ACirc}\, p + a \cdot \mathrm{ACirc}\, q = \mathrm{ACirc}(a \cdot (p + q))$.

(66) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $a \cdot \mathrm{ACirc}\, p + b \cdot \mathrm{ACirc}\, q = \mathrm{ACirc}(a \cdot p + b \cdot q)$.

Let us consider K, n. Observe that $0_K^{n \times n}$ is anti-circular.

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