Integral of Complex-Valued Measurable Function

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Summary. In this article, we formalized the notion of the integral of a complex-valued function considered as a sum of its real and imaginary parts. Then we defined the measurability and integrability in this context, and proved the linearity and several other basic properties of complex-valued measurable functions. The set of properties showed in this paper is based on [15], where the case of real-valued measurable functions is considered.

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The notation and terminology used here are introduced in the following papers: [17], [1], [11], [18], [6], [19], [7], [2], [12], [14], [16], [5], [4], [3], [9], [10], [13], [8], and [15].

1. Definitions for Complex-valued Functions

One can prove the following proposition

(1) For all real numbers a, b holds $\overline{\mathbb{R}}(a) + \overline{\mathbb{R}}(b) = a + b$ and $-\overline{\mathbb{R}}(a) = -a$ and $\overline{\mathbb{R}}(a) - \overline{\mathbb{R}}(b) = a - b$ and $\overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b) = a \cdot b$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{C}$. The functor $\Re(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 1) $\operatorname{dom} \Re(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom} \Re(f)$ holds $\Re(f)(x) = \Re(f(x))$.

The functor $\Im(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\operatorname{dom} \Im(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom} \Im(f)$ holds $\Im(f)(x) = \Im(f(x))$.

2. The Measurability of Complex-valued Functions

For simplicity, we use the following convention: X is a non empty set, Y is a set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\mathbb{C}$, r is a real number, c is a complex number, and E, A, B are elements of S.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\mathbb{C}$, and let E be an element of S. We say that f is measurable on E if and only if:

(Def. 3) $\Re(f)$ is measurable on E and $\Im(f)$ is measurable on E.

One can prove the following propositions:

(2) $r \Re(f) = \Re(r\,f)$ and $r \Im(f) = \Im(r\,f)$.

(3) $\Re(c\,f) = \Re(c)\,\Re(f) - \Im(c)\,\Im(f)$ and $\Im(c\,f) = \Im(c)\,\Re(f) + \Re(c)\,\Im(f)$.

(4) $-\Im(f) = \Re(i\,f)$ and $\Re(f) = \Im(i\,f)$.

(5) $\Re(f + g) = \Re(f) + \Re(g)$ and $\Im(f + g) = \Im(f) + \Im(g)$.

(6) $\Re(f - g) = \Re(f) - \Re(g)$ and $\Im(f - g) = \Im(f) - \Im(g)$.

(7) $\Re(f){\upharpoonright}A = \Re(f{\upharpoonright}A)$ and $\Im(f){\upharpoonright}A = \Im(f{\upharpoonright}A)$.

(8) $f = \Re(f) + i\,\Im(f)$.

(9)   If $B \subseteq A$ and f is measurable on A, then f is measurable on B.

(10)   If f is measurable on A and f is measurable on B, then f is measurable on $A \cup B$.

(11)   If f is measurable on A and g is measurable on A, then $f+g$ is measurable on A.

(12)   If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $f - g$ is measurable on A.

(13)   If $Y \subseteq \operatorname{dom}(f+g)$, then $\operatorname{dom}(f{\upharpoonright}Y + g{\upharpoonright}Y) = Y$ and $(f+g){\upharpoonright}Y = f{\upharpoonright}Y + g{\upharpoonright}Y$.

(14)   If f is measurable on B and $A = \operatorname{dom} f \cap B$, then $f{\upharpoonright}B$ is measurable on A.

(15)   If $\operatorname{dom} f$, $\operatorname{dom} g \in S$, then $\operatorname{dom}(f + g) \in S$.

(16)   If $\operatorname{dom} f = A$, then f is measurable on B iff f is measurable on $A \cap B$.

(17)   If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $c\,f$ is measurable on A.

(18)   Given an element A of S such that $\operatorname{dom} f = A$. Let c be a complex number and B be an element of S. If f is measurable on B, then $c\,f$ is measurable on B.

3. The Integral of a Complex-valued Measurable Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. We say that f is integrable on M if and only if:

(Def. 4) $\Re(f)$ is integrable on M and $\Im(f)$ is integrable on M.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. Let us assume that f is integrable on M. The functor $\int f \, dM$ yielding a complex number is defined by:

(Def. 5)   There exist real numbers R, I such that $R = \int \Re(f) \, dM$ and $I = \int \Im(f) \, dM$ and $\int f \, dM = R + I \cdot i$.

We now state several propositions:

(19)   Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $f \restriction A$ is integrable on M.

(20)   Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\mathbb{R}$, and E, A be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $f \restriction A$ is integrable on M.

(21)   Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $f \restriction A$ is integrable on M and $\int f \restriction A \, dM = 0$.

(22)   If $E = \operatorname{dom} f$ and f is integrable on M and $M(A) = 0$, then $\int f \restriction (E \setminus A) \, dM = \int f \, dM$.

(23)   If f is integrable on M, then $f \restriction A$ is integrable on M.

(24)   If f is integrable on M and A misses B, then $\int f \restriction (A \cup B) \, dM = \int f \restriction A \, dM + \int f \restriction B \, dM$.

(25)   If f is integrable on M and $B = \operatorname{dom} f \setminus A$, then $f \restriction A$ is integrable on M and $\int f \, dM = \int f \restriction A \, dM + \int f \restriction B \, dM$.

Let k be a real number, let X be a non empty set, and let f be a partial function from X to $\mathbb{R}$. The functor f^k yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 6) $\operatorname{dom}(f^k) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Let us consider X. Observe that there exists a partial function from X to $\mathbb{R}$ which is non-negative.

Let k be a non negative real number, let us consider X, and let f be a non-negative partial function from X to $\mathbb{R}$. Observe that f^k is non-negative.

We now state a number of propositions:

(26) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative and $0 \le k$, then f^k is non-negative.

(27) Let x be a set, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative, then $f(x)^{\frac{1}{2}} = \sqrt{f(x)}$.

(28) For every partial function f from X to $\mathbb{R}$ and for every real number a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \operatorname{LE-dom}(f, a) = A \setminus A \cap \operatorname{GTE-dom}(f, a)$.

(29) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. Suppose f is non-negative and $0 \le k$ and $E \subseteq \operatorname{dom} f$ and f is measurable on E. Then f^k is measurable on E.

(30) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $|f|$ is measurable on A.

(31) Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(32) If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(33) If f is integrable on M and g is integrable on M, then $f + g$ is integrable on M.

(34) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then $f - g$ is integrable on M.

(35) If f is integrable on M and g is integrable on M, then $f - g$ is integrable on M.

(36) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f + g \, dM = \int f{\upharpoonright}E \, dM + \int g{\upharpoonright}E \, dM$.

(37) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f - g \, dM = \int f{\upharpoonright}E \, dM + \int (-g){\upharpoonright}E \, dM$.

(38) If f is integrable on M, then $r f$ is integrable on M and $\int r f \, dM = r \cdot \int f \, dM$.

(39) If f is integrable on M, then $i f$ is integrable on M and $\int i f \, dM = i \cdot \int f \, dM$.

(40) If f is integrable on M, then $c f$ is integrable on M and $\int c f \, dM = c \cdot \int f \, dM$.

(41) For every partial function f from X to $\mathbb{R}$ and for all Y, r holds $(r f){\upharpoonright}Y = r (f{\upharpoonright}Y)$.

(42) Let f, g be partial functions from X to $\mathbb{R}$. Suppose that

(i)   there exists an element A of S such that $A = \operatorname{dom} f \cap \operatorname{dom} g$ and f is measurable on A and g is measurable on A,

(ii) f is integrable on M,

(iii) g is integrable on M, and

(iv) $g - f$ is non-negative.

Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f{\upharpoonright}E \, \mathrm{d}M \leq \int g{\upharpoonright}E \, \mathrm{d}M$.

(43)   Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is integrable on M. Then $|\int f \, \mathrm{d}M| \leq \int |f| \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{C}$, and let B be an element of S. The functor $\int_B f \, \mathrm{d}M$ yields a complex number and is defined by:

(Def. 7) $\int\limits_B f \, \mathrm{d}M = \int f{\upharpoonright}B \, \mathrm{d}M$.

Next we state two propositions:

(44)   Suppose f is integrable on M and g is integrable on M and $B \subseteq \operatorname{dom}(f + g)$. Then $f + g$ is integrable on M and $\int\limits_B f + g \, \mathrm{d}M = \int\limits_B f \, \mathrm{d}M + \int\limits_B g \, \mathrm{d}M$.

(45)   If f is integrable on M and f is measurable on B, then $\int\limits_B c\, f \, \mathrm{d}M = c \cdot \int\limits_B f \, \mathrm{d}M$.

4. Several Properties of Real-valued Measurable Functions

In the sequel f denotes a partial function from X to $\mathbb{R}$ and a denotes a real number.

One can prove the following four propositions:

(46)   If $A \subseteq \operatorname{dom} f$, then $A \cap \operatorname{GTE-dom}(f, a) = A \setminus A \cap \operatorname{LE-dom}(f, a)$.

(47)   If $A \subseteq \operatorname{dom} f$, then $A \cap \operatorname{GT-dom}(f, a) = A \setminus A \cap \operatorname{LEQ-dom}(f, a)$.

(48)   If $A \subseteq \operatorname{dom} f$, then $A \cap \operatorname{LEQ-dom}(f, a) = A \setminus A \cap \operatorname{GT-dom}(f, a)$.

(49) $A \cap \operatorname{EQ-dom}(f, a) = A \cap \operatorname{GTE-dom}(f, a) \cap \operatorname{LEQ-dom}(f, a)$.

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