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Jordan Matrix Decomposition

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Summary. In this paper I present the Jordan Matrix Decomposition Theorem which states that an arbitrary square matrix M over an algebraically closed field can be decomposed into the form

$$M = SJS^{-1}$$

where S is an invertible matrix and J is a matrix in a Jordan canonical form, i.e. a special type of block diagonal matrix in which each block consists of Jordan blocks (see [13]).

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The terminology and notation used here are introduced in the following articles: [11], [2], [3], [12], [34], [7], [10], [8], [4], [28], [33], [30], [18], [6], [14], [15], [36], [23], [37], [35], [9], [29], [32], [31], [5], [19], [24], [22], [17], [1], [21], [20], [16], [25], [27], and [26].

1. Jordan Blocks

We follow the rules: i, j, m, n, k denote natural numbers, K denotes a field, and a, λ denote elements of K.

Let us consider K, λ, n. The Jordan block of λ and n yields a matrix over K and is defined by the conditions (Def. 1).

(Def. 1)(i) len (the Jordan block of λ and n) $= n$,

(ii) width (the Jordan block of λ and n) $= n$, and

(iii) for all i, j such that $\langle i, j\rangle \in$ the indices of the Jordan block of λ and n holds if $i = j$, then (the Jordan block of λ and $n)_{i,j} = \lambda$ and if $i + 1 = j$, then (the Jordan block of λ and $n)_{i,j} = \mathbf{1}_K$ and if $i \neq j$ and $i + 1 \neq j$, then (the Jordan block of λ and $n)_{i,j} = 0_K$.

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Let us consider K, λ, n. Then the Jordan block of λ and n is an upper triangular matrix over K of dimension n.

The following propositions are true:

(1) The diagonal of the Jordan block of λ and $n = n \mapsto \lambda$.

(2) Det (the Jordan block of λ and n) $= \mathrm{power}_K(\lambda, n)$.

(3) The Jordan block of λ and n is invertible iff $n = 0$ or $\lambda \neq 0_K$.

(4) If $i \in \mathrm{Seg}\, n$ and $i \neq n$, then Line(the Jordan block of λ and n, i) $= \lambda \cdot \mathrm{Line}(I_K^{n \times n}, i) + \mathrm{Line}(I_K^{n \times n}, i+1)$.

(5) Line(the Jordan block of λ and n, n) $= \lambda \cdot \mathrm{Line}(I_K^{n \times n}, n)$.

(6) Let F be an element of (the carrier of $K)^n$ such that $i \in \mathrm{Seg}\, n$. Then

(i) if $i \neq n$, then Line(the Jordan block of λ and n, i) $\cdot F = \lambda \cdot F_i + F_{i+1}$, and

(ii) if $i = n$, then Line(the Jordan block of λ and n, i) $\cdot F = \lambda \cdot F_i$.

(7) Let F be an element of (the carrier of $K)^n$ such that $i \in \mathrm{Seg}\, n$. Then

(i) if $i = 1$, then (the Jordan block of λ and $n)_{\square,i} \cdot F = \lambda \cdot F_i$, and

(ii) if $i \neq 1$, then (the Jordan block of λ and $n)_{\square,i} \cdot F = \lambda \cdot F_i + F_{i-1}$.

(8) Suppose $\lambda \neq 0_K$. Then there exists a square matrix M over K of dimension n such that

(i) (the Jordan block of λ and $n)^{\smile} = M$, and

(ii) for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i > j$, then $M_{i,j} = 0_K$ and if $i \leq j$, then $M_{i,j} = -\mathrm{power}_K(-\lambda^{-1}, (j -' i) + 1)$.

(9) (The Jordan block of λ and n) $+ a \cdot I_K^{n \times n}$ = the Jordan block of $\lambda + a$ and n.

2. Finite Sequences of Jordan Blocks

Let us consider K and let G be a finite sequence of elements of ((the carrier of $K)^*)^*$. We say that G is Jordan-block-yielding if and only if:

(Def. 2) For every i such that $i \in \mathrm{dom}\, G$ there exist λ, n such that $G(i)$ = the Jordan block of λ and n.

Let us consider K. Observe that there exists a finite sequence of elements of ((the carrier of $K)^*)^*$ which is Jordan-block-yielding.

Let us consider K. One can verify that every finite sequence of elements of ((the carrier of $K)^*)^*$ which is Jordan-block-yielding is also square-matrix-yielding.

Let us consider K. A finite sequence of Jordan blocks of K is a Jordan-block-yielding finite sequence of elements of ((the carrier of $K)^*)^*$.

Let us consider K, λ. A finite sequence of Jordan blocks of K is said to be a finite sequence of Jordan blocks of λ and K if:

(Def. 3) For every i such that $i \in \operatorname{dom} \mathrm{it}$ there exists n such that $\mathrm{it}(i) =$ the Jordan block of λ and n.

Next we state two propositions:

(10) $\emptyset$ is a finite sequence of Jordan blocks of λ and K.

(11) $\langle$the Jordan block of λ and $n\rangle$ is a finite sequence of Jordan blocks of λ and K.

Let us consider K, λ. Observe that there exists a finite sequence of Jordan blocks of λ and K which is non-empty.

Let us consider K. Note that there exists a finite sequence of Jordan blocks of K which is non-empty.

Next we state the proposition

(12) Let J be a finite sequence of Jordan blocks of λ and K. Then $J \oplus \operatorname{len} J \mapsto a \bullet I_K^{\operatorname{Len} J \times \operatorname{Len} J}$ is a finite sequence of Jordan blocks of $\lambda + a$ and K.

Let us consider K and let J_1, J_2 be fininte sequences of Jordan blocks of K. Then $J_1 \frown J_2$ is a finite sequence of Jordan blocks of K.

Let us consider K, let J be a finite sequence of Jordan blocks of K, and let us consider n. Then $J{\restriction}n$ is a finite sequence of Jordan blocks of K. Then $J_{\lfloor n}$ is a finite sequence of Jordan blocks of K.

Let us consider K, λ and let J_1, J_2 be finite sequences of Jordan blocks of λ and K. Then $J_1 \frown J_2$ is a finite sequence of Jordan blocks of λ and K.

Let us consider K, λ, let J be a finite sequence of Jordan blocks of λ and K, and let us consider n. Then $J{\restriction}n$ is a finite sequence of Jordan blocks of λ and K. Then $J_{\lfloor n}$ is a finite sequence of Jordan blocks of λ and K.

3. Nilpotent Transformations

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a function from V into V. We say that f is nilpotent if and only if:

(Def. 4) There exists n such that for every vector v of V holds $f^n(v) = 0_V$.

We now state the proposition

(13) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a function from V into V. Then f is nilpotent if and only if there exists n such that $f^n = \mathrm{ZeroMap}(V, V)$.

Let K be a double loop structure and let V be a non empty vector space structure over K. Observe that there exists a function from V into V which is nilpotent.

Let R be a ring and let V be a left module over R. Observe that there exists a function from V into V which is nilpotent and linear.

Next we state the proposition

(14) Let V be a vector space over K and f be a linear transformation from V to V. Then $f\restriction \ker f^n$ is a nilpotent linear transformation from $\ker f^n$ to $\ker f^n$.

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a nilpotent function from V into V. The degree of nilpotence of f yielding a natural number is defined by the conditions (Def. 5).

(Def. 5)(i) $f^{\text{the degree of nilpotence of } f} = \mathrm{ZeroMap}(V, V)$, and

(ii) for every k such that $f^k = \mathrm{ZeroMap}(V, V)$ holds the degree of nilpotence of $f \leq k$.

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a nilpotent function from V into V. We introduce $\deg f$ as a synonym of the degree of nilpotence of f.

One can prove the following propositions:

(15) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a nilpotent function from V into V. Then $\deg f = 0$ if and only if $\Omega_V = \{0_V\}$.

(16) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a nilpotent function from V into V. Then there exists a vector v of V such that for every i such that $i < \deg f$ holds $f^i(v) \neq 0_V$.

(17) Let K be a field, V be a vector space over K, W be a subspace of V, and f be a nilpotent function from V into V. Suppose $f\restriction W$ is a function from W into W. Then $f\restriction W$ is a nilpotent function from W into W.

(18) Let K be a field, V be a vector space over K, W be a subspace of V, f be a nilpotent linear transformation from V to V, and f_1 be a nilpotent function from $\mathrm{im}(f^n)$ into $\mathrm{im}(f^n)$. If $f_1 = f\restriction \mathrm{im}(f^n)$ and $n \leq \deg f$, then $\deg f_1 + n = \deg f$.

For simplicity, we adopt the following convention: V_1, V_2 denote finite dimensional vector spaces over K, W_1, W_2 denote subspaces of V_1, U_1, U_2 denote subspaces of V_2, b_1 denotes an ordered basis of V_1, B_1 denotes a finite sequence of elements of V_1, b_2 denotes an ordered basis of V_2, B_2 denotes a finite sequence of elements of V_2, b_3 denotes an ordered basis of W_1, b_4 denotes an ordered basis of W_2, B_3 denotes a finite sequence of elements of U_1, and B_4 denotes a finite sequence of elements of U_2.

Next we state a number of propositions:

(19) Let M be a matrix over K of dimension $\mathrm{len}\, b_1 \times \mathrm{len}\, B_2$, M_1 be a matrix over K of dimension $\mathrm{len}\, b_3 \times \mathrm{len}\, B_3$, and M_2 be a matrix over K of dimension $\mathrm{len}\, b_4 \times \mathrm{len}\, B_4$ such that $b_1 = b_3 \frown b_4$ and $B_2 = B_3 \frown B_4$ and $M =$ the 0_K-block diagonal of $\langle M_1, M_2 \rangle$ and $\mathrm{width}\, M_1 = \mathrm{len}\, B_3$ and $\mathrm{width}\, M_2 = \mathrm{len}\, B_4$. Then

(i) if $i \in \operatorname{dom} b_3$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = (\mathrm{Mx2Tran}(M_1, b_3, B_3))((b_3)_i)$, and

(ii) if $i \in \operatorname{dom} b_4$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_{i+\operatorname{len} b_3}) = (\mathrm{Mx2Tran}(M_2, b_4, B_4))((b_4)_i)$.

(20) Let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$ and F be a finite sequence of matrices over K. Suppose $M =$ the 0_K-block diagonal of F. Let given i, m. Suppose $i \in \operatorname{dom} b_1$ and $m = \min(\operatorname{Len} F, i)$. Then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \sum \mathrm{lmlt}(\mathrm{Line}(F(m), i -' \sum \mathrm{Len}(F \restriction (m -' 1))), (B_2 \restriction \sum \mathrm{Width}(F \restriction m))_{\lfloor \sum \mathrm{Width}(F \restriction (m -' 1))})$ and $\operatorname{len}((B_2 \restriction \sum \mathrm{Width}(F \restriction m))_{\lfloor \sum \mathrm{Width}(F \restriction (m -' 1))}) = \operatorname{width} F(m)$.

(21) If $\operatorname{len} B_1 \in \operatorname{dom} B_1$, then $\sum \mathrm{lmlt}(\mathrm{Line}(\text{the Jordan block of } \lambda \text{ and } \operatorname{len} B_1, \operatorname{len} B_1), B_1) = \lambda \cdot (B_1)_{\operatorname{len} B_1}$.

(22) If $i \in \operatorname{dom} B_1$ and $i \neq \operatorname{len} B_1$, then $\sum \mathrm{lmlt}(\mathrm{Line}(\text{the Jordan block of } \lambda \text{ and } \operatorname{len} B_1, i), B_1) = \lambda \cdot (B_1)_i + (B_1)_{i+1}$.

(23) Let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. Suppose $M =$ the Jordan block of λ and n. Let given i such that $i \in \operatorname{dom} b_1$. Then

(i) if $i = \operatorname{len} b_1$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and

(ii) if $i \neq \operatorname{len} b_1$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.

(24) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. Suppose $M =$ the 0_K-block diagonal of J. Let given i, m such that $i \in \operatorname{dom} b_1$ and $m = \min(\operatorname{Len} J, i)$. Then

(i) if $i = \sum \mathrm{Len}(J \restriction m)$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and

(ii) if $i \neq \sum \mathrm{Len}(J \restriction m)$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.

(25) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J. Let given m. If for every i such that $i \in \operatorname{dom} J$ holds $\operatorname{len} J(i) \leq m$, then $(\mathrm{Mx2Tran}(M, b_1, b_1))^m = \mathrm{ZeroMap}(V_1, V_1)$.

(26) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J. Then $\mathrm{Mx2Tran}(M, b_1, b_1)$ is nilpotent if and only if $\operatorname{len} b_1 = 0$ or $\lambda = 0_K$.

(27) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J and $\operatorname{len} b_1 > 0$. Let F be a nilpotent function from V_1 into V_1. Suppose $F = \mathrm{Mx2Tran}(M, b_1, b_1)$. Then there exists i such that $i \in \operatorname{dom} J$ and $\operatorname{len} J(i) = \deg F$ and for every i such that $i \in \operatorname{dom} J$ holds $\operatorname{len} J(i) \leq \deg F$.

(28) Let given V_1, V_2, b_1, b_2, λ. Suppose $\operatorname{len} b_1 = \operatorname{len} b_2$. Let F be a linear

transformation from V_1 to V_2. Suppose that for every i such that $i \in \operatorname{dom} b_1$ holds $F((b_1)_i) = \lambda \cdot (b_2)_i$ or $i+1 \in \operatorname{dom} b_1$ and $F((b_1)_i) = \lambda \cdot (b_2)_i + (b_2)_{i+1}$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K such that $\mathrm{AutMt}(F, b_1, b_2) =$ the 0_K-block diagonal of J.

(29) Let V_1 be a finite dimensional vector space over K and F be a nilpotent linear transformation from V_1 to V_1. Then there exists a non-empty finite sequence J of Jordan blocks of 0_K and K and there exists an ordered basis b_1 of V_1 such that $\mathrm{AutMt}(F, b_1, b_1) =$ the 0_K-block diagonal of J.

(30) Let V be a vector space over K, F be a linear transformation from V to V, V_1 be a finite dimensional subspace of V, and F_1 be a linear transformation from V_1 to V_1. Suppose $V_1 = \ker (F + (-\lambda) \cdot \mathrm{id}_V)^n$ and $F \restriction V_1 = F_1$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K and there exists an ordered basis b_1 of V_1 such that $\mathrm{AutMt}(F_1, b_1, b_1) =$ the 0_K-block diagonal of J.

4. The Main Theorem

The following two propositions are true:

(31) Let K be an algebraic-closed field, V be a non trivial finite dimensional vector space over K, and F be a linear transformation from V to V. Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists an ordered basis b_1 of V such that

(i) $\mathrm{AutMt}(F, b_1, b_1) =$ the 0_K-block diagonal of J, and

(ii) for every scalar λ of K holds λ is an eigenvalue of F iff there exists i such that $i \in \operatorname{dom} J$ and $J(i) =$ the Jordan block of λ and $\operatorname{len} J(i)$.

(32) Let K be an algebraic-closed field and M be a square matrix over K of dimension n. Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists a square matrix P over K of dimension n such that $\sum \mathrm{Len}\, J = n$ and P is invertible and $M = P \cdot$ the 0_K-block diagonal of $J \cdot P^{\smile}$.

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