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Eigenvalues of a Linear Transformation

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Summary. The article presents well known facts about eigenvalues of linear transformation of a vector space (see [13]). I formalize main dependencies between eigenvalues and the diagram of the matrix of a linear transformation over a finite-dimensional vector space. Finally, I formalize the subspace $\bigcup_{i=0}^{\infty} \mathrm{Ker}(f-\lambda I)^i$ called a generalized eigenspace for the eigenvalue λ and show its basic properties.

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The articles [11], [33], [2], [3], [12], [34], [8], [10], [9], [5], [31], [27], [15], [7], [14], [32], [35], [25], [30], [29], [28], [26], [6], [22], [16], [23], [20], [1], [19], [4], [21], [17], [18], and [24] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following convention: i, j, m, n denote natural numbers, K denotes a field, and a denotes an element of K.

Next we state several propositions:

(1) Let A, B be matrices over K, n_1 be an element of $\mathbb{N}^n$, and m_1 be an element of $\mathbb{N}^m$. If $\mathrm{rng}\, n_1 \times \mathrm{rng}\, m_1 \subseteq$ the indices of A, then $\mathrm{Segm}(A+B, n_1, m_1) = \mathrm{Segm}(A, n_1, m_1) + \mathrm{Segm}(B, n_1, m_1)$.

(2) For every without zero finite subset P of $\mathbb{N}$ such that $P \subseteq \mathrm{Seg}\, n$ holds $\mathrm{Segm}(I_K^{n\times n}, P, P) = I_K^{\mathrm{card}\, P\times \mathrm{card}\, P}$.

(3) Let A, B be matrices over K and P, Q be without zero finite subsets of $\mathbb{N}$. If $P\times Q \subseteq$ the indices of A, then $\mathrm{Segm}(A+B, P, Q) = \mathrm{Segm}(A, P, Q) + \mathrm{Segm}(B, P, Q)$.

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(4) For all square matrices A, B over K of dimension n such that $i, j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(A+B, i, j) = \operatorname{Delete}(A, i, j) + \operatorname{Delete}(B, i, j)$.

(5) For every square matrix A over K of dimension n such that $i, j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(a \cdot A, i, j) = a \cdot \operatorname{Delete}(A, i, j)$.

(6) If $i \in \operatorname{Seg} n$, then $\operatorname{Delete}(I_K^{n \times n}, i, i) = I_K^{(n -' 1) \times (n -' 1)}$.

(7) Let A, B be square matrices over K of dimension n. Then there exists a polynomial P of K such that $\operatorname{len} P \leq n+1$ and for every element x of K holds $\operatorname{eval}(P, x) = \operatorname{Det}(A + x \cdot B)$.

(8) Let A be a square matrix over K of dimension n. Then there exists a polynomial P of K such that $\operatorname{len} P = n+1$ and for every element x of K holds $\operatorname{eval}(P, x) = \operatorname{Det}(A + x \cdot I_K^{n \times n})$.

Let us consider K. Observe that there exists a vector space over K which is non trivial and finite dimensional.

2. Maps with Eigenvalues

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, and let I_1 be a function from V into V. We say that I_1 has eigenvalues if and only if:

(Def. 1) There exists a vector v of V and there exists a scalar a of R such that $v \neq 0_V$ and $I_1(v) = a \cdot v$.

For simplicity, we follow the rules: V denotes a non trivial vector space over K, V_1, V_2 denote vector spaces over K, f denotes a linear transformation from V_1 to V_1, v, w denote vectors of V, v_1 denotes a vector of V_1, and L denotes a scalar of K.

Let us consider K, V. One can verify that there exists a linear transformation from V to V which has eigenvalues.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, and let f be a function from V into V. Let us assume that f has eigenvalues. An element of R is called an eigenvalue of f if:

(Def. 2) There exists a vector v of V such that $v \neq 0_V$ and $f(v) = \mathrm{it} \cdot v$.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, let f be a function from V into V, and let L be a scalar of R. Let us assume that f has eigenvalues and L is an eigenvalue of f. A vector of V is called an eigenvector of f and L if:

(Def. 3) $f(\mathrm{it}) = L \cdot \mathrm{it}$.

We now state several propositions:

(9) Let given a. Suppose $a \neq 0_K$. Let f be a function from V into V with eigenvalues and L be an eigenvalue of f. Then

(i) $a \cdot f$ has eigenvalues,

(ii) $a \cdot L$ is an eigenvalue of $a \cdot f$, and

(iii) w is an eigenvector of f and L iff w is an eigenvector of $a \cdot f$ and $a \cdot L$.

(10) Let f_1, f_2 be functions from V into V with eigenvalues and L_1, L_2 be scalars of K. Suppose that

(i) L_1 is an eigenvalue of f_1,

(ii) L_2 is an eigenvalue of f_2, and

(iii) there exists v such that v is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 and $v \neq 0_V$.

Then

(iv) $f_1 + f_2$ has eigenvalues,

(v) $L_1 + L_2$ is an eigenvalue of $f_1 + f_2$, and

(vi) for every w such that w is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 holds w is an eigenvector of $f_1 + f_2$ and $L_1 + L_2$.

(11) id_V has eigenvalues and $\mathbf{1}_K$ is an eigenvalue of id_V and every v is an eigenvector of id_V and $\mathbf{1}_K$.

(12) For every eigenvalue L of id_V holds $L = \mathbf{1}_K$.

(13) If $\ker f$ is non trivial, then f has eigenvalues and 0_K is an eigenvalue of f.

(14) f has eigenvalues and L is an eigenvalue of f iff $\ker f + (-L) \cdot \mathrm{id}_{(V_1)}$ is non trivial.

(15) Let V_1 be a finite dimensional vector space over K, b_1, b_1' be ordered bases of V_1, and f be a linear transformation from V_1 to V_1. Then f has eigenvalues and L is an eigenvalue of f if and only if $\mathrm{Det}\,\mathrm{AutEqMt}(f + (-L) \cdot \mathrm{id}_{(V_1)}, b_1, b_1') = 0_K$.

(16) Let K be an algebraic-closed field and V_1 be a non trivial finite dimensional vector space over K. Then every linear transformation from V_1 to V_1 has eigenvalues.

(17) Let given f, L. Suppose f has eigenvalues and L is an eigenvalue of f. Then v_1 is an eigenvector of f and L if and only if $v_1 \in \ker f + (-L) \cdot \mathrm{id}_{(V_1)}$.

Let S be a 1-sorted structure, let F be a function from S into S, and let n be a natural number. The functor F^n yields a function from S into S and is defined as follows:

(Def. 4) For every element F' of the semigroup of functions onto the carrier of S such that $F' = F$ holds $F^n = \prod(n \mapsto F')$.

In the sequel S denotes a 1-sorted structure and F denotes a function from S into S.

Next we state several propositions:

(18) $F^0 = \mathrm{id}_S$.

(19) $F^1 = F$.

(20) $F^{i+j} = F^i \cdot F^j$.

(21) For all elements s_1, s_2 of S and for all n, m such that $F^m(s_1) = s_2$ and $F^n(s_2) = s_2$ holds $F^{m+i\cdot n}(s_1) = s_2$.

(22) Let K be an add-associative right zeroed right complementable Abelian associative well unital distributive non empty double loop structure, V_1 be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K, W be a subspace of V_1, f be a function from V_1 into V_1, and f_3 be a function from W into W. If $f_3 = f \restriction W$, then $f^n \restriction W = {f_3}^n$.

Let us consider K, V_1, let f be a linear transformation from V_1 to V_1, and let n be a natural number. Then f^n is a linear transformation from V_1 to V_1.

We now state the proposition

(23) If $f^i(v_1) = 0_{(V_1)}$, then $f^{i+j}(v_1) = 0_{(V_1)}$.

3. Generalized Eigenspace of a Linear Transformation

Let us consider K, V_1, f. The functor UnionKers f yielding a strict subspace of V_1 is defined by:

(Def. 5) The carrier of UnionKers $f = \{v; v$ ranges over vectors of $V_1\colon \bigvee_n \ f^n(v) = 0_{(V_1)}\}$.

We now state a number of propositions:

(24) $v_1 \in$ UnionKers f iff there exists n such that $f^n(v_1) = 0_{(V_1)}$.

(25) $\ker f^i$ is a subspace of UnionKers f.

(26) $\ker f^i$ is a subspace of $\ker f^{i+j}$.

(27) Let V be a finite dimensional vector space over K and f be a linear transformation from V to V. Then there exists n such that UnionKers $f = \ker f^n$.

(28) $f \restriction \ker f^n$ is a linear transformation from $\ker f^n$ to $\ker f^n$.

(29) $f \restriction \ker (f + L \cdot \mathrm{id}_{(V_1)})^n$ is a linear transformation from $\ker (f + L \cdot \mathrm{id}_{(V_1)})^n$ to $\ker (f + L \cdot \mathrm{id}_{(V_1)})^n$.

(30) $f \restriction$ UnionKers f is a linear transformation from UnionKers f to UnionKers f.

(31) $f \restriction \mathrm{UnionKers}(f + L \cdot \mathrm{id}_{(V_1)})$ is a linear transformation from $\mathrm{UnionKers}(f + L \cdot \mathrm{id}_{(V_1)})$ to $\mathrm{UnionKers}(f + L \cdot \mathrm{id}_{(V_1)})$.

(32) $f \restriction \mathrm{im}(f^n)$ is a linear transformation from $\mathrm{im}(f^n)$ to $\mathrm{im}(f^n)$.

(33) $f \restriction \mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$ is a linear transformation from $\mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$ to $\mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$.

(34) If UnionKers $f = \ker f^n$, then $\ker f^n \cap \mathrm{im}(f^n) = \mathbf{0}_{(V_1)}$.

(35) Let V be a finite dimensional vector space over K, f be a linear transformation from V to V, and given n. If UnionKers $f = \ker f^n$, then V is the direct sum of $\ker f^n$ and $\mathrm{im}(f^n)$.

(36) For every linear complement I of UnionKers f holds $f \restriction I$ is one-to-one.

(37) Let I be a linear complement of UnionKers$(f + (-L) \cdot \mathrm{id}_{(V_1)})$ and f_4 be a linear transformation from I to I. If $f_4 = f \restriction I$, then for every vector v of I such that $f_4(v) = L \cdot v$ holds $v = 0_{(V_1)}$.

(38) Suppose $n \geq 1$. Then there exists a linear transformation h from V_1 to V_1 such that $(f + L \cdot \mathrm{id}_{(V_1)})^n = f \cdot h + (L \cdot \mathrm{id}_{(V_1)})^n$ and for every i holds $f^i \cdot h = h \cdot f^i$.

(39) Let L_1, L_2 be scalars of K. Suppose f has eigenvalues and $L_1 \neq L_2$ and L_1 is an eigenvalue of f and L_2 is an eigenvalue of f. Let I be a linear complement of UnionKers$(f + (-L_1) \cdot \mathrm{id}_{(V_1)})$ and f_4 be a linear transformation from I to I. Suppose $f_4 = f \restriction I$. Then f_4 has eigenvalues and L_1 is not an eigenvalue of f_4 and L_2 is an eigenvalue of f_4 and UnionKers$(f + (-L_2) \cdot \mathrm{id}_{(V_1)})$ is a subspace of I.

(40) Let U be a finite subset of V_1. Suppose U is linearly independent. Let u be a vector of V_1. Suppose $u \in U$. Let L be a linear combination of $U \setminus \{u\}$. Then $\overline{\overline{U}} = \overline{\overline{(U \setminus \{u\}) \cup \{u + \sum L\}}}$ and $(U \setminus \{u\}) \cup \{u + \sum L\}$ is linearly independent.

(41) Let A be a subset of V_1, L be a linear combination of V_2, and f be a linear transformation from V_1 to V_2. Suppose the support of $L \subseteq f^\circ A$. Then there exists a linear combination M of A such that $f(\sum M) = \sum L$.

(42) Let f be a linear transformation from V_1 to V_2, A be a subset of V_1, and B be a subset of V_2. If $f^\circ A = B$, then f°(the carrier of $\mathrm{Lin}(A)$) = the carrier of $\mathrm{Lin}(B)$.

(43) Let L be a linear combination of V_1, F be a finite sequence of elements of V_1, and f be a linear transformation from V_1 to V_2. Suppose $f \restriction$((the support of L) $\cap \operatorname{rng} F$) is one-to-one and $\operatorname{rng} F \subseteq$ the support of L. Then there exists a linear combination L_3 of V_2 such that

(i) the support of $L_3 = f^\circ$((the support of L) $\cap \operatorname{rng} F$),

(ii) $f \cdot (L\,F) = L_3\,(f \cdot F)$, and

(iii) for every v_1 such that $v_1 \in$ (the support of L) $\cap \operatorname{rng} F$ holds $L(v_1) = L_3(f(v_1))$.

(44) Let A, B be subsets of V_1 and L be a linear combination of V_1. Suppose the support of $L \subseteq A \cup B$ and $\sum L = 0_{(V_1)}$. Let f be a linear function from V_1 into V_2. Suppose $f \restriction B$ is one-to-one and $f^\circ B$ is a linearly independent subset of V_2 and $f^\circ A \subseteq \{0_{(V_2)}\}$. Then the support of $L \subseteq A$.

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