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Block Diagonal Matrices

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Summary. In this paper I present basic properties of block diagonal matrices over a set. In my approach the finite sequence of matrices in a block diagonal matrix is not restricted to square matrices. Moreover, the off-diagonal blocks need not be zero matrices, but also with another arbitrary fixed value.

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The papers [19], [1], [2], [6], [7], [3], [17], [16], [12], [5], [8], [9], [20], [13], [18], [21], [4], [14], [15], [11], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: i, j, m, n, k denote natural numbers, x denotes a set, K denotes a field, a, a_1, a_2 denote elements of K, D denotes a non empty set, d, d_1, d_2 denote elements of D, M, M_1, M_2 denote matrices over D, A, A_1, A_2, B_1, B_2 denote matrices over K, and f, g denote finite sequences of elements of $\mathbb{N}$.

One can prove the following propositions:

(1) Let K be a non empty additive loop structure and f_1, f_2, g_1, g_2 be finite sequences of elements of K. If $\operatorname{len} f_1 = \operatorname{len} f_2$, then $(f_1 + f_2) \frown (g_1 + g_2) = f_1 \frown g_1 + f_2 \frown g_2$.

(2) For all finite sequences f, g of elements of D such that $i \in \operatorname{dom} f$ holds $(f \frown g)_{\restriction i} = (f_{\restriction i}) \frown g$.

(3) For all finite sequences f, g of elements of D such that $i \in \operatorname{dom} g$ holds $(f \frown g)_{\restriction i + \operatorname{len} f} = f \frown (g_{\restriction i})$.

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(4) If $i \in \mathrm{Seg}(n+1)$, then $((n+1) \mapsto d)_{\restriction i} = n \mapsto d$.

(5) $\prod(n \mapsto a) = \mathrm{power}_K(a, n)$.

Let us consider f and let i be a natural number. Let us assume that $i \in \mathrm{Seg}(\sum f)$. The functor $\min(f, i)$ yielding an element of $\mathbb{N}$ is defined by:

(Def. 1) $i \leq \sum f \restriction \min(f, i)$ and $\min(f, i) \in \mathrm{dom}\, f$ and for every j such that $i \leq \sum f \restriction j$ holds $\min(f, i) \leq j$.

One can prove the following propositions:

(6) If $i \in \mathrm{dom}\, f$ and $f(i) \neq 0$, then $\min(f, \sum f \restriction i) = i$.

(7) If $i \in \mathrm{Seg}(\sum f)$, then $\min(f, i) -' 1 = \min(f, i) - 1$ and $\sum f \restriction (\min(f, i) -' 1) < i$.

(8) If $i \in \mathrm{Seg}(\sum f)$, then $\min(f \frown g, i) = \min(f, i)$.

(9) If $i \in \mathrm{Seg}((\sum f) + \sum g) \setminus \mathrm{Seg}(\sum f)$, then $\min(f \frown g, i) = \min(g, i -' \sum f) + \mathrm{len}\, f$ and $i -' \sum f = i - \sum f$.

(10) If $i \in \mathrm{dom}\, f$ and $j \in \mathrm{Seg}(f_i)$, then $j + \sum f \restriction (i -' 1) \in \mathrm{Seg}(\sum f \restriction i)$ and $\min(f, j + \sum f \restriction (i -' 1)) = i$.

(11) For all i, j such that $i \leq \mathrm{len}\, f$ and $j \leq \mathrm{len}\, f$ and $\sum f \restriction i = \sum f \restriction j$ and if $i \in \mathrm{dom}\, f$, then $f(i) \neq 0$ and if $j \in \mathrm{dom}\, f$, then $f(j) \neq 0$ holds $i = j$.

2. Finite Sequences of Matrices

Let us consider D and let F be a finite sequence of elements of $(D^*)^*$. We say that F is matrix-yielding if and only if:

(Def. 2) For every i such that $i \in \mathrm{dom}\, F$ holds $F(i)$ is a matrix over D.

Let us consider D. Observe that there exists a finite sequence of elements of $(D^*)^*$ which is matrix-yielding.

Let us consider D. A finite sequence of matrices over D is a matrix-yielding finite sequence of elements of $(D^*)^*$.

Let us consider K. A finite sequence of matrices over K is a matrix-yielding finite sequence of elements of ((the carrier of $K)^*)^*$.

We now state the proposition

(12) $\emptyset$ is a finite sequence of matrices over D.

We adopt the following rules: F, F_1, F_2 are finite sequences of matrices over D and G, G', G_1, G_2 are finite sequences of matrices over K.

Let us consider D, F, x. Then $F(x)$ is a matrix over D.

Let us consider D, F_1, F_2. Then $F_1 \frown F_2$ is a finite sequence of matrices over D.

Let us consider D, M_1. Then $\langle M_1 \rangle$ is a finite sequence of matrices over D. Let us consider M_2. Then $\langle M_1, M_2 \rangle$ is a finite sequence of matrices over D.

Let us consider D, F, n. Then $F{\restriction}n$ is a finite sequence of matrices over D. Then $F_{\lfloor n}$ is a finite sequence of matrices over D.

3. Sequences of Sizes of Matrices in a Finite Sequence

Let us consider D, F. The functor $\operatorname{Len} F$ yielding a finite sequence of elements of $\mathbb{N}$ is defined as follows:

(Def. 3) $\operatorname{dom}\operatorname{Len} F = \operatorname{dom} F$ and for every i such that $i \in \operatorname{dom}\operatorname{Len} F$ holds $(\operatorname{Len} F)(i) = \operatorname{len} F(i)$.

The functor $\operatorname{Width} F$ yields a finite sequence of elements of $\mathbb{N}$ and is defined by:

(Def. 4) $\operatorname{dom}\operatorname{Width} F = \operatorname{dom} F$ and for every i such that $i \in \operatorname{dom}\operatorname{Width} F$ holds $(\operatorname{Width} F)(i) = \operatorname{width} F(i)$.

Let us consider D, F. Then $\operatorname{Len} F$ is an element of $\mathbb{N}^{\operatorname{len} F}$. Then $\operatorname{Width} F$ is an element of $\mathbb{N}^{\operatorname{len} F}$.

The following propositions are true:

(13) If $\sum \operatorname{Len} F = 0$, then $\sum \operatorname{Width} F = 0$.

(14) $\operatorname{Len}(F_1 \frown F_2) = (\operatorname{Len} F_1) \frown \operatorname{Len} F_2$.

(15) $\operatorname{Len}\langle M\rangle = \langle \operatorname{len} M\rangle$.

(16) $\sum \operatorname{Len}\langle M_1, M_2\rangle = \operatorname{len} M_1 + \operatorname{len} M_2$.

(17) $\operatorname{Len}(F_1{\restriction}n) = \operatorname{Len} F_1{\restriction}n$.

(18) $\operatorname{Width}(F_1 \frown F_2) = (\operatorname{Width} F_1) \frown \operatorname{Width} F_2$.

(19) $\operatorname{Width}\langle M\rangle = \langle \operatorname{width} M\rangle$.

(20) $\sum \operatorname{Width}\langle M_1, M_2\rangle = \operatorname{width} M_1 + \operatorname{width} M_2$.

(21) $\operatorname{Width}(F_1{\restriction}n) = \operatorname{Width} F_1{\restriction}n$.

4. Block Diagonal Matrices

Let us consider D, let d be an element of D, and let F be a finite sequence of matrices over D. The d-block diagonal of F is a matrix over D and is defined by the conditions (Def. 5).

(Def. 5)(i) len (the d-block diagonal of F) $= \sum \operatorname{Len} F$,

(ii) width (the d-block diagonal of F) $= \sum \operatorname{Width} F$, and

(iii) for all i, j such that $\langle i, j\rangle \in$ the indices of the d-block diagonal of F holds if $j \leq \sum \operatorname{Width} F{\restriction}(\min(\operatorname{Len} F, i) -' 1)$ or $j > \sum \operatorname{Width} F{\restriction}\min(\operatorname{Len} F, i)$, then (the d-block diagonal of $F)_{i,j} = d$ and if $\sum \operatorname{Width} F{\restriction}(\min(\operatorname{Len} F, i) -' 1) < j \leq \sum \operatorname{Width} F{\restriction}\min(\operatorname{Len} F, i)$, then (the d-block diagonal of $F)_{i,j} = F(\min(\operatorname{Len} F, i))_{i -' \sum \operatorname{Len} F{\restriction}(\min(\operatorname{Len} F, i) -' 1),\, j -' \sum \operatorname{Width} F{\restriction}(\min(\operatorname{Len} F, i) -' 1)}$.

Let us consider D, let d be an element of D, and let F be a finite sequence of matrices over D. Then the d-block diagonal of F is a matrix over D of dimension $\sum \mathrm{Len}\, F \times \sum \mathrm{Width}\, F$.

Next we state a number of propositions:

(22) For every finite sequence F of matrices over D such that $F = \emptyset$ holds the d-block diagonal of $F = \emptyset$.

(23) Let M be a matrix over D of dimension $\sum \mathrm{Len}\langle M_1, M_2\rangle \times \sum \mathrm{Width}\langle M_1, M_2\rangle$. Then $M =$ the d-block diagonal of $\langle M_1, M_2\rangle$ if and only if for every i holds if $i \in \mathrm{dom}\, M_1$, then $\mathrm{Line}(M, i) = \mathrm{Line}(M_1, i) \frown (\mathrm{width}\, M_2 \mapsto d)$ and if $i \in \mathrm{dom}\, M_2$, then $\mathrm{Line}(M, i + \mathrm{len}\, M_1) = (\mathrm{width}\, M_1 \mapsto d) \frown \mathrm{Line}(M_2, i)$.

(24) Let M be a matrix over D of dimension $\sum \mathrm{Len}\langle M_1, M_2\rangle \times \sum \mathrm{Width}\langle M_1, M_2\rangle$. Then $M =$ the d-block diagonal of $\langle M_1, M_2\rangle$ if and only if for every i holds if $i \in \mathrm{Seg}\,\mathrm{width}\, M_1$, then $M_{\square,i} = ((M_1)_{\square,i}) \frown (\mathrm{len}\, M_2 \mapsto d)$ and if $i \in \mathrm{Seg}\,\mathrm{width}\, M_2$, then $M_{\square,i+\mathrm{width}\, M_1} = (\mathrm{len}\, M_1 \mapsto d) \frown ((M_2)_{\square,i})$.

(25) The indices of the d_1-block diagonal of F_1 is a subset of the indices of the d_2-block diagonal of $F_1 \frown F_2$.

(26) Suppose $\langle i, j\rangle \in$ the indices of the d-block diagonal of F_1. Then (the d-block diagonal of $F_1)_{i,j} =$ (the d-block diagonal of $F_1 \frown F_2)_{i,j}$.

(27) $\langle i, j\rangle \in$ the indices of the d_1-block diagonal of F_2 if and only if $i > 0$ and $j > 0$ and $\langle i + \sum \mathrm{Len}\, F_1, j + \sum \mathrm{Width}\, F_1\rangle \in$ the indices of the d_2-block diagonal of $F_1 \frown F_2$.

(28) Suppose $\langle i, j\rangle \in$ the indices of the d-block diagonal of F_2. Then (the d-block diagonal of $F_2)_{i,j} =$ (the d-block diagonal of $F_1 \frown F_2)_{i+\sum \mathrm{Len}\, F_1, j+\sum \mathrm{Width}\, F_1}$.

(29) Suppose $\langle i, j\rangle \in$ the indices of the d-block diagonal of $F_1 \frown F_2$ but $i \leq \sum \mathrm{Len}\, F_1$ and $j > \sum \mathrm{Width}\, F_1$ or $i > \sum \mathrm{Len}\, F_1$ and $j \leq \sum \mathrm{Width}\, F_1$. Then (the d-block diagonal of $F_1 \frown F_2)_{i,j} = d$.

(30) Let given i, j, k. Suppose $i \in \mathrm{dom}\, F$ and $\langle j, k\rangle \in$ the indices of $F(i)$. Then

(i) $\langle j + \sum \mathrm{Len}\, F{\restriction}(i -' 1), k + \sum \mathrm{Width}\, F{\restriction}(i -' 1)\rangle \in$ the indices of the d-block diagonal of F, and

(ii) $F(i)_{j,k} =$ (the d-block diagonal of $F)_{j+\sum \mathrm{Len}\, F\restriction(i-'1), k+\sum \mathrm{Width}\, F\restriction(i-'1)}$.

(31) If $i \in \mathrm{dom}\, F$, then $F(i) = \mathrm{Segm}($the d-block diagonal of $F, \mathrm{Seg}(\sum \mathrm{Len}\, F{\restriction}i) \setminus \mathrm{Seg}(\sum \mathrm{Len}\, F{\restriction}(i -' 1)), \mathrm{Seg}(\sum \mathrm{Width}\, F{\restriction}i) \setminus \mathrm{Seg}(\sum \mathrm{Width}\, F{\restriction}(i -' 1)))$.

(32) $M = \mathrm{Segm}($the d-block diagonal of $\langle M\rangle \frown F, \mathrm{Seg}\,\mathrm{len}\, M, \mathrm{Seg}\,\mathrm{width}\, M)$.

(33) $M = \mathrm{Segm}($the d-block diagonal of $F \frown \langle M\rangle, \mathrm{Seg}(\mathrm{len}\, M + \sum \mathrm{Len}\, F) \setminus \mathrm{Seg}(\sum \mathrm{Len}\, F), \mathrm{Seg}(\mathrm{width}\, M + \sum \mathrm{Width}\, F) \setminus \mathrm{Seg}(\sum \mathrm{Width}\, F))$.

(34) The d-block diagonal of $\langle M\rangle = M$.

(35) The d-block diagonal of $F_1 \frown F_2 =$ the d-block diagonal of $\langle$the d-block diagonal of $F_1\rangle \frown F_2$.

(36) The d-block diagonal of $F_1 \frown F_2 =$ the d-block diagonal of $F_1 \frown \langle$the d-block diagonal of $F_2\rangle$.

(37) If $i \in \mathrm{Seg}(\sum \mathrm{Len}\, F)$ and $m = \min(\mathrm{Len}\, F, i)$, then Line(the d-block diagonal of F, i) $= ((\sum \mathrm{Width}(F{\restriction}(m -' 1))) \mapsto d) \frown \mathrm{Line}(F(m), i -' \sum \mathrm{Len}(F{\restriction}(m -' 1))) \frown (((\sum \mathrm{Width}\, F) -' \sum \mathrm{Width}(F{\restriction}m)) \mapsto d)$.

(38) If $i \in \mathrm{Seg}(\sum \mathrm{Width}\, F)$ and $m = \min(\mathrm{Width}\, F, i)$, then (the d-block diagonal of $F)_{\square,i} = ((\sum \mathrm{Len}(F{\restriction}(m -' 1))) \mapsto d) \frown (F(m)_{\square,i-'\sum \mathrm{Width}(F{\restriction}(m-'1))}) \frown (((\sum \mathrm{Len}\, F) -' \sum \mathrm{Len}(F{\restriction}m)) \mapsto d)$.

(39) Let M_1, M_2, N_1, N_2 be matrices over D. Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, N_1$ and $\mathrm{width}\, M_1 = \mathrm{width}\, N_1$ and $\mathrm{len}\, M_2 = \mathrm{len}\, N_2$ and $\mathrm{width}\, M_2 = \mathrm{width}\, N_2$ and the d_1-block diagonal of $\langle M_1, M_2\rangle =$ the d_2-block diagonal of $\langle N_1, N_2\rangle$. Then $M_1 = N_1$ and $M_2 = N_2$.

(40) Suppose $M = \emptyset$. Then

(i) the d-block diagonal of $F \frown \langle M\rangle =$ the d-block diagonal of F, and

(ii) the d-block diagonal of $\langle M\rangle \frown F =$ the d-block diagonal of F.

(41) Suppose $i \in \mathrm{dom}\, A$ and $\mathrm{width}\, A = \mathrm{width}$ (the deleting of i-row in A). Then the deleting of i-row in the a-block diagonal of $\langle A\rangle \frown G =$ the a-block diagonal of $\langle$the deleting of i-row in $A\rangle \frown G$.

(42) Suppose $i \in \mathrm{dom}\, A$ and $\mathrm{width}\, A = \mathrm{width}$ (the deleting of i-row in A). Then the deleting of $(\sum \mathrm{Len}\, G)+i$-row in the a-block diagonal of $G \frown \langle A\rangle =$ the a-block diagonal of $G \frown \langle$the deleting of i-row in $A\rangle$.

(43) Suppose $i \in \mathrm{Seg}\, \mathrm{width}\, A$. Then the deleting of i-column in the a-block diagonal of $\langle A\rangle \frown G =$ the a-block diagonal of $\langle$the deleting of i-column in $A\rangle \frown G$.

(44) Suppose $i \in \mathrm{Seg}\, \mathrm{width}\, A$. Then the deleting of $i + \sum \mathrm{Width}\, G$-column in the a-block diagonal of $G \frown \langle A\rangle =$ the a-block diagonal of $G \frown \langle$the deleting of i-column in $A\rangle$.

Let us consider D and let F be a finite sequence of elements of $(D^*)^*$. We say that F is square-matrix-yielding if and only if:

(Def. 6) For every i such that $i \in \mathrm{dom}\, F$ there exists n such that $F(i)$ is a square matrix over D of dimension n.

Let us consider D. One can verify that there exists a finite sequence of elements of $(D^*)^*$ which is square-matrix-yielding.

Let us consider D. Observe that every finite sequence of elements of $(D^*)^*$ which is square-matrix-yielding is also matrix-yielding.

Let us consider D. A finite sequence of square-matrices over D is a square-matrix-yielding finite sequence of elements of $(D^*)^*$.

Let us consider K. A finite sequence of square-matrices over K is a square-matrix-yielding finite sequence of elements of ((the carrier of $K)^*)^*$.

We use the following convention: S, S_1, S_2 denote finite sequences of square-matrices over D and R, R_1, R_2 denote finite sequences of square-matrices over K.

One can prove the following proposition

(45) $\emptyset$ is a finite sequence of square-matrices over D.

Let us consider D, S, x. Then $S(x)$ is a square matrix over D of dimension $\operatorname{len} S(x)$.

Let us consider D, S_1, S_2. Then $S_1 \frown S_2$ is a finite sequence of square-matrices over D.

Let us consider D, n and let M_1 be a square matrix over D of dimension n. Then $\langle M_1 \rangle$ is a finite sequence of square-matrices over D.

Let us consider D, n, m, let M_1 be a square matrix over D of dimension n, and let M_2 be a square matrix over D of dimension m. Then $\langle M_1, M_2 \rangle$ is a finite sequence of square-matrices over D.

Let us consider D, S, n. Then $S \restriction n$ is a finite sequence of square-matrices over D. Then $S_{\lfloor n}$ is a finite sequence of square-matrices over D.

The following proposition is true

(46) $\operatorname{Len} S = \operatorname{Width} S$.

Let us consider D, let d be an element of D, and let S be a finite sequence of square-matrices over D. Then the d-block diagonal of S is a square matrix over D of dimension $\sum \operatorname{Len} S$.

One can prove the following propositions:

(47) Let A be a square matrix over K of dimension n. Suppose $i \in \operatorname{dom} A$ and $j \in \operatorname{Seg} n$. Then the deleting of i-row and j-column in the a-block diagonal of $\langle A \rangle \frown R =$ the a-block diagonal of $\langle$the deleting of i-row and j-column in $A\rangle \frown R$.

(48) Let A be a square matrix over K of dimension n. Suppose $i \in \operatorname{dom} A$ and $j \in \operatorname{Seg} n$. Then the deleting of $i + \sum \operatorname{Len} R$-row and $j + \sum \operatorname{Len} R$-column in the a-block diagonal of $R \frown \langle A \rangle =$ the a-block diagonal of $R \frown \langle$the deleting of i-row and j-column in $A\rangle$.

Let us consider K, R. The functor $\operatorname{Det} R$ yielding a finite sequence of elements of K is defined as follows:

(Def. 7) $\operatorname{dom} \operatorname{Det} R = \operatorname{dom} R$ and for every i such that $i \in \operatorname{dom} \operatorname{Det} R$ holds $(\operatorname{Det} R)(i) = \operatorname{Det} R(i)$.

Let us consider K, R. Then $\operatorname{Det} R$ is an element of (the carrier of $K)^{\operatorname{len} R}$.

In the sequel N denotes a square matrix over K of dimension n and N_1 denotes a square matrix over K of dimension m.

The following propositions are true:

(49) $\mathrm{Det}\langle N\rangle = \langle \mathrm{Det}\, N\rangle$.

(50) $\mathrm{Det}(R_1 \frown R_2) = (\mathrm{Det}\, R_1) \frown \mathrm{Det}\, R_2$.

(51) $\mathrm{Det}(R{\restriction}n) = \mathrm{Det}\, R{\restriction}n$.

(52) Det (the 0_K-block diagonal of $\langle N, N_1\rangle$) $= \mathrm{Det}\, N \cdot \mathrm{Det}\, N_1$.

(53) Det (the 0_K-block diagonal of R) $= \prod \mathrm{Det}\, R$.

(54) If $\mathrm{len}\, A_1 \neq \mathrm{width}\, A_1$ and $N =$ the 0_K-block diagonal of $\langle A_1, A_2\rangle$, then $\mathrm{Det}\, N = 0_K$.

(55) Suppose $\mathrm{Len}\, G \neq \mathrm{Width}\, G$. Let M be a square matrix over K of dimension n. If $M =$ the 0_K-block diagonal of G, then $\mathrm{Det}\, M = 0_K$.

5. An Example of a Finite Sequence of Matrices

Let us consider K and let f be a finite sequence of elements of $\mathbb{N}$. The functor $I_K^{f\times f}$ yielding a finite sequence of square-matrices over K is defined by:

(Def. 8) $\mathrm{dom}(I_K^{f\times f}) = \mathrm{dom}\, f$ and for every i such that $i \in \mathrm{dom}(I_K^{f\times f})$ holds $I_K^{f\times f}(i) = I_K^{f(i)\times f(i)}$.

The following propositions are true:

(56) $\mathrm{Len}(I_K^{f\times f}) = f$ and $\mathrm{Width}(I_K^{f\times f}) = f$.

(57) For every element i of $\mathbb{N}$ holds $I_K^{\langle i\rangle\times\langle i\rangle} = \langle I_K^{i\times i}\rangle$.

(58) $I_K^{(f\frown g)\times(f\frown g)} = (I_K^{f\times f}) \frown I_K^{g\times g}$.

(59) $I_K^{(f\restriction n)\times(f\restriction n)} = I_K^{f\times f}{\restriction}n$.

(60) The 0_K-block diagonal of $\langle I_K^{i\times i}, I_K^{j\times j}\rangle = I_K^{(i+j)\times(i+j)}$.

(61) The 0_K-block diagonal of $I_K^{f\times f} = I_K^{(\sum f)\times(\sum f)}$.

In the sequel p, p_1 are finite sequences of elements of K.

6. Operations on a Finite Sequence of Matrices

Let us consider K, G, p. The functor $p \bullet G$ yielding a finite sequence of matrices over K is defined as follows:

(Def. 9) $\mathrm{dom}(p \bullet G) = \mathrm{dom}\, G$ and for every i such that $i \in \mathrm{dom}(p \bullet G)$ holds $(p \bullet G)(i) = p_i \cdot G(i)$.

Let us consider K and let us consider R, p. Then $p \bullet R$ is a finite sequence of square-matrices over K.

The following propositions are true:

(62) $\mathrm{Len}(p \bullet G) = \mathrm{Len}\, G$ and $\mathrm{Width}(p \bullet G) = \mathrm{Width}\, G$.

(63) $p \bullet \langle A\rangle = \langle p_1 \cdot A\rangle$.

(64) If $\mathrm{len}\, G = \mathrm{len}\, p$ and $\mathrm{len}\, G_1 \leq \mathrm{len}\, p_1$, then $p \frown p_1 \bullet G \frown G_1 = (p \bullet G) \frown (p_1 \bullet G_1)$.

(65) $a \cdot$the a_1-block diagonal of $G =$ the $(a \cdot a_1)$-block diagonal of $\operatorname{len} G \mapsto a \bullet G$.

Let us consider K and let G_1, G_2 be finite sequences of matrices over K. The functor $G_1 \oplus G_2$ yields a finite sequence of matrices over K and is defined by:

(Def. 10) $\operatorname{dom}(G_1 \oplus G_2) = \operatorname{dom} G_1$ and for every i such that $i \in \operatorname{dom}(G_1 \oplus G_2)$ holds $(G_1 \oplus G_2)(i) = G_1(i) + G_2(i)$.

Let us consider K and let us consider R, G. Then $R \oplus G$ is a finite sequence of square-matrices over K.

The following propositions are true:

(66) $\operatorname{Len}(G_1 \oplus G_2) = \operatorname{Len} G_1$ and $\operatorname{Width}(G_1 \oplus G_2) = \operatorname{Width} G_1$.

(67) If $\operatorname{len} G = \operatorname{len} G'$, then $G \frown G_1 \oplus G' \frown G_2 = (G \oplus G') \frown (G_1 \oplus G_2)$.

(68) $\langle A \rangle \oplus G = \langle A + G(1) \rangle$.

(69) $\langle A_1 \rangle \oplus \langle A_2 \rangle = \langle A_1 + A_2 \rangle$.

(70) $\langle A_1, B_1 \rangle \oplus \langle A_2, B_2 \rangle = \langle A_1 + A_2, B_1 + B_2 \rangle$.

(71) Suppose $\operatorname{len} A_1 = \operatorname{len} B_1$ and $\operatorname{len} A_2 = \operatorname{len} B_2$ and $\operatorname{width} A_1 = \operatorname{width} B_1$ and $\operatorname{width} A_2 = \operatorname{width} B_2$. Then (the a_1-block diagonal of $\langle A_1, A_2 \rangle$) + (the a_2-block diagonal of $\langle B_1, B_2 \rangle$) = the $(a_1 + a_2)$-block diagonal of $\langle A_1, A_2 \rangle \oplus \langle B_1, B_2 \rangle$.

(72) Suppose $\operatorname{Len} R_1 = \operatorname{Len} R_2$ and $\operatorname{Width} R_1 = \operatorname{Width} R_2$. Then (the a_1-block diagonal of R_1) + (the a_2-block diagonal of R_2) = the $(a_1 + a_2)$-block diagonal of $R_1 \oplus R_2$.

Let us consider K and let G_1, G_2 be finite sequences of matrices over K. The functor $G_1\, G_2$ yielding a finite sequence of matrices over K is defined by:

(Def. 11) $\operatorname{dom}(G_1\, G_2) = \operatorname{dom} G_1$ and for every i such that $i \in \operatorname{dom}(G_1\, G_2)$ holds $(G_1\, G_2)(i) = G_1(i) \cdot G_2(i)$.

Next we state several propositions:

(73) If $\operatorname{Width} G_1 = \operatorname{Len} G_2$, then $\operatorname{Len}(G_1\, G_2) = \operatorname{Len} G_1$ and $\operatorname{Width}(G_1\, G_2) = \operatorname{Width} G_2$.

(74) If $\operatorname{len} G = \operatorname{len} G'$, then $(G \frown G_1)\,(G' \frown G_2) = (G\, G') \frown (G_1\, G_2)$.

(75) $\langle A \rangle\, G = \langle A \cdot G(1) \rangle$.

(76) $\langle A_1 \rangle\, \langle A_2 \rangle = \langle A_1 \cdot A_2 \rangle$.

(77) $\langle A_1, B_1 \rangle\, \langle A_2, B_2 \rangle = \langle A_1 \cdot A_2, B_1 \cdot B_2 \rangle$.

(78) Suppose $\operatorname{width} A_1 = \operatorname{len} B_1$ and $\operatorname{width} A_2 = \operatorname{len} B_2$. Then (the 0_K-block diagonal of $\langle A_1, A_2 \rangle$) $\cdot$ (the 0_K-block diagonal of $\langle B_1, B_2 \rangle$) = the 0_K-block diagonal of $\langle A_1, A_2 \rangle\, \langle B_1, B_2 \rangle$.

(79) If $\operatorname{Width} R_1 = \operatorname{Len} R_2$, then (the 0_K-block diagonal of R_1) $\cdot$ (the 0_K-block diagonal of R_2) = the 0_K-block diagonal of $R_1\, R_2$.

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