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General Theory of Quasi-Commutative BCI-algebras

Tao Sun
Qingdao University of Science and Technology
China

Weibo Pan
Qingdao University of Science and Technology
China

Chenglong Wu
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Summary. It is known that commutative BCK-algebras form a variety, but BCK-algebras do not [4]. Therefore H. Yutani introduced the notion of quasi-commutative BCK-algebras. In this article we first present the notion and general theory of quasi-commutative BCI-algebras. Then we discuss the reduction of the type of quasi-commutative BCK-algebras and some special classes of quasi-commutative BCI-algebras.

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The articles [7], [2], [3], [1], [5], and [6] provide the terminology and notation for this paper.

Let X be a BCI-algebra, let x, y be elements of X, and let m, n be elements of $\mathbb{N}$. The functor Polynom(m,n,x,y) yields an element of X and is defined as follows:

(Def. 1) Polynom$(m,n,x,y) = ((x \setminus (x \setminus y))^{m+1} \setminus (y \setminus x))^n$.

We adopt the following convention: X denotes a BCI-algebra, x, y, z denote elements of X, and i, j, k, l, m, n denote elements of $\mathbb{N}$.

One can prove the following propositions:

(1) If $x \leq y \leq z$, then $x \leq z$.

(2) If $x \leq y \leq x$, then $x = y$.

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(3) For every BCK-algebra X and for all elements x, y of X holds $x \setminus y \leq x$ and $(x \setminus y)^{n+1} \leq (x \setminus y)^n$.

(4) For every BCK-algebra X and for every element x of X holds $(0_X \setminus x)^n = 0_X$.

(5) For every BCK-algebra X and for all elements x, y of X such that $m \geq n$ holds $(x \setminus y)^m \leq (x \setminus y)^n$.

(6) Let X be a BCK-algebra and x, y be elements of X. Suppose $m > n$ and $(x \setminus y)^n = (x \setminus y)^m$. Let k be an element of $\mathbb{N}$. If $k \geq n$, then $(x \setminus y)^n = (x \setminus y)^k$.

(7) $\mathrm{Polynom}(0, 0, x, y) = x \setminus (x \setminus y)$.

(8) $\mathrm{Polynom}(m, n, x, y) = ((\mathrm{Polynom}(0, 0, x, y) \setminus (x \setminus y))^m \setminus (y \setminus x))^n$.

(9) $\mathrm{Polynom}(m + 1, n, x, y) = \mathrm{Polynom}(m, n, x, y) \setminus (x \setminus y)$.

(10) $\mathrm{Polynom}(m, n + 1, x, y) = \mathrm{Polynom}(m, n, x, y) \setminus (y \setminus x)$.

(11) $\mathrm{Polynom}(n + 1, n + 1, y, x) \leq \mathrm{Polynom}(n, n + 1, x, y)$.

(12) $\mathrm{Polynom}(n, n + 1, x, y) \leq \mathrm{Polynom}(n, n, y, x)$.

Let X be a BCI-algebra. We say that X is quasi-commutative if and only if:

(Def. 2) There exist elements i, j, m, n of $\mathbb{N}$ such that for all elements x, y of X holds $\mathrm{Polynom}(i, j, x, y) = \mathrm{Polynom}(m, n, y, x)$.

Let us observe that BCI-EXAMPLE is quasi-commutative.

One can check that there exists a BCI-algebra which is quasi-commutative.

Let i, j, m, n be elements of $\mathbb{N}$. A BCI-algebra is called a BCI-algebra commutating with i, j and m, n if:

(Def. 3) For all elements x, y of it holds $\mathrm{Polynom}(i, j, x, y) = \mathrm{Polynom}(m, n, y, x)$.

One can prove the following propositions:

(13) X is a BCI-algebra commutating with i, j and m, n if and only if X is a BCI-algebra commutating with m, n and i, j.

(14) Let X be a BCI-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCI-algebra commutating with $i + k$, j and m, $n + k$.

(15) Let X be a BCI-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCI-algebra commutating with i, $j + k$ and $m + k$, n.

One can verify that there exists a BCK-algebra which is quasi-commutative.

Let i, j, m, n be elements of $\mathbb{N}$. One can check that there exists a BCI-algebra commutating with i, j and m, n which is BCK-5.

Let i, j, m, n be elements of $\mathbb{N}$. A BCK-algebra commutating with i, j and m, n is BCK-5 BCI-algebra commutating with i, j and m, n.

One can prove the following propositions:

(16) X is a BCK-algebra commutating with i, j and m, n if and only if X is a BCK-algebra commutating with m, n and i, j.

(17) Let X be a BCK-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCK-algebra commutating with $i+k$, j and m, $n+k$.

(18) Let X be a BCK-algebra commutating with i, j and m, n and k be an element of $\mathbb{N}$. Then X is a BCK-algebra commutating with i, $j+k$ and $m+k$, n.

(19) For every BCK-algebra X commutating with i, j and m, n and for all elements x, y of X holds $(x \setminus y)^{i+1} = (x \setminus y)^{n+1}$.

(20) For every BCK-algebra X commutating with i, j and m, n and for all elements x, y of X holds $(x \setminus y)^{j+1} = (x \setminus y)^{m+1}$.

(21) Every BCK-algebra commutating with i, j and m, n is a BCK-algebra commutating with i, j and j, n.

(22) Every BCK-algebra commutating with i, j and m, n is a BCK-algebra commutating with n, j and m, n.

Let us consider i, j, m, n. The functor $\min(i,j,m,n)$ yielding an extended real number is defined as follows:

(Def. 4) $\min(i,j,m,n) = \min(\min(i,j), \min(m,n))$.

The functor $\max(i,j,m,n)$ yielding an extended real number is defined by:

(Def. 5) $\max(i,j,m,n) = \max(\max(i,j), \max(m,n))$.

Next we state a number of propositions:

(23) $\min(i,j,m,n) = i$ or $\min(i,j,m,n) = j$ or $\min(i,j,m,n) = m$ or $\min(i,j,m,n) = n$.

(24) $\max(i,j,m,n) = i$ or $\max(i,j,m,n) = j$ or $\max(i,j,m,n) = m$ or $\max(i,j,m,n) = n$.

(25) If $i = \min(i,j,m,n)$, then $i \leq j$ and $i \leq m$ and $i \leq n$.

(26) $\max(i,j,m,n) \geq i$ and $\max(i,j,m,n) \geq j$ and $\max(i,j,m,n) \geq m$ and $\max(i,j,m,n) \geq n$.

(27) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i,j,m,n)$. If $i = j$, then X is a BCK-algebra commutating with i, i and i, i.

(28) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i,j,m,n)$. Suppose $i < j$ and $i < n$. Then X is a BCK-algebra commutating with i, $i+1$ and i, $i+1$.

(29) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i,j,m,n)$. Suppose $i < j$ and $i = n$ and $i = m$. Then X is a BCK-algebra commutating with i, i and i, i.

(30) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i = n$ and $i < m < j$. Then X is a BCK-algebra commutating with i, $m + 1$ and m, i.

(31) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = \min(i, j, m, n)$. Suppose $i < j$ and $i = n$ and $j \leq m$. Then X is a BCK-algebra commutating with i, j and j, i.

(32) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $l \geq j$ and $k \geq n$. Then X is a BCK-algebra commutating with k, l and l, k.

(33) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $k \geq \max(i, j, m, n)$. Then X is a BCK-algebra commutating with k, k and k, k.

(34) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i \leq m$ and $j \leq n$. Then X is a BCK-algebra commutating with i, j and i, j.

(35) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i \leq m$ and $i < n$. Then X is a BCK-algebra commutating with i, j and i, $i + 1$.

(36) If X is a BCI-algebra commutating with i, j and $j + k$, $i + k$, then X is a BCK-algebra.

(37) X is a BCI-algebra commutating with 0, 0 and 0, 0 if and only if X is a BCK-algebra commutating with 0, 0 and 0, 0.

(38) X is a commutative BCK-algebra iff X is a BCI-algebra commutating with 0, 0 and 0, 0.

Let X be a BCI-algebra. We introduce p-Semisimple-part X as a synonym of AtomSet X.

In the sequel B, P are non empty subsets of X.

One can prove the following propositions:

(39) For every BCI-algebra X such that B = BCK-part X and P = p-Semisimple-part X holds $B \cap P = \{0_X\}$.

(40) For every BCI-algebra X such that $P = p$-Semisimple-part X holds X is a BCK-algebra iff $P = \{0_X\}$.

(41) For every BCI-algebra X such that B = BCK-part X holds X is a p-semisimple BCI-algebra iff $B = \{0_X\}$.

(42) If X is a p-semisimple BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(43) Suppose X is a p-semisimple BCI-algebra. Then X is a BCI-algebra commutating with $n + j$, n and m, $m + j + 1$.

(44) Suppose X is an associative BCI-algebra. Then X is a BCI-algebra commutating with 0, 1 and 0, 0 and a BCI-algebra commutating with 1, 0 and 0, 0.

(45) Suppose X is a weakly-positive-implicative BCI-algebra. Then X is a BCI-algebra commutating with 0, 1 and 1, 1.

(46) If X is a positive-implicative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 1, 1.

(47) If X is an implicative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(48) If X is an alternative BCI-algebra, then X is a BCI-algebra commutating with 0, 1 and 0, 0.

(49) X is a BCK-positive-implicative BCK-algebra if and only if X is a BCK-algebra commutating with 0, 1 and 0, 1.

(50) X is a BCK-implicative BCK-algebra iff X is a BCK-algebra commutating with 1, 0 and 0, 0.

One can check that every BCK-algebra which is BCK-implicative is also commutative and every BCK-algebra which is BCK-implicative is also BCK-positive-implicative.

The following propositions are true:

(51) X is a BCK-algebra commutating with 1, 0 and 0, 0 if and only if X is a BCK-algebra commutating with 0, 0 and 0, 0 and a BCK-algebra commutating with 0, 1 and 0, 1.

(52) Let X be a quasi-commutative BCK-algebra. Then X is a BCK-algebra commutating with 0, 1 and 0, 1 if and only if for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y$.

(53) Let X be a quasi-commutative BCK-algebra. Then X is a BCK-algebra commutating with n, $n+1$ and n, $n+1$ if and only if for all elements x, y of X holds $(x \setminus y)^{n+1} = (x \setminus y)^{n+2}$.

(54) If X is a BCI-algebra commutating with 0, 1 and 0, 0, then X is a BCI-commutative BCI-algebra.

(55) If X is a BCI-algebra commutating with n, 0 and m, m, then X is a BCI-commutative BCI-algebra.

(56) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $j = 0$ and $m > 0$. Then X is a BCK-algebra commutating with 0, 0 and 0, 0.

(57) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $m = 0$ and $j > 0$. Then X is a BCK-algebra commutating with 0, 1 and 0, 1.

(58) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $n = 0$ and $i \neq 0$. Then X is a BCK-algebra commutating with 0, 0 and 0, 0.

(59) Let X be a BCK-algebra commutating with i, j and m, n. Suppose $i = 0$ and $n \neq 0$. Then X is a BCK-algebra commutating with 0, 1 and 0, 1.

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