FORMALIZED MATHEMATICS
Vol. 16, No. 2, Pages 91–96, 2008
DOI: 10.2478/v10037-008-0013-3

Helly Property for Subtrees[1]

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Summary. We prove, following [5, p. 92], that any family of subtrees of a finite tree satisfies the Helly property.

MML identifier: `HELLY`, version: `7.8.09 4.97.1001`

The articles [12], [4], [10], [3], [2], [1], [11], [9], [8], [7], and [6] provide the notation and terminology for this paper.

1. General Preliminaries

One can prove the following proposition

(1) For every non empty finite sequence p holds $\langle p(1) \rangle \frown p = p$.

Let p, q be finite sequences. The functor $\operatorname{maxPrefix}(p, q)$ yields a finite sequence and is defined by:

(Def. 1) $\operatorname{maxPrefix}(p, q) \preceq p$ and $\operatorname{maxPrefix}(p, q) \preceq q$ and for every finite sequence r such that $r \preceq p$ and $r \preceq q$ holds $r \preceq \operatorname{maxPrefix}(p, q)$.

Let us observe that the functor $\operatorname{maxPrefix}(p, q)$ is commutative.

Next we state several propositions:

(2) For all finite sequences p, q holds $p \preceq q$ iff $\operatorname{maxPrefix}(p, q) = p$.

(3) For all finite sequences p, q holds $\operatorname{len} \operatorname{maxPrefix}(p, q) \leq \operatorname{len} p$.

(4) For every non empty finite sequence p holds $\langle p(1) \rangle \preceq p$.

(5) For all non empty finite sequences p, q such that $p(1) = q(1)$ holds $1 \leq \operatorname{len} \operatorname{maxPrefix}(p, q)$.

[1]This work has been partially supported by the NSERC grant OGP 9207.

ISSN 1426–2630(p), 1898-9934(e)

(6) For all finite sequences p, q and for every natural number j such that $j \leq \operatorname{len} \operatorname{maxPrefix}(p, q)$ holds $(\operatorname{maxPrefix}(p, q))(j) = p(j)$.

(7) For all finite sequences p, q and for every natural number j such that $j \leq \operatorname{len} \operatorname{maxPrefix}(p, q)$ holds $p(j) = q(j)$.

(8) For all finite sequences p, q holds $p \not\preceq q$ iff $\operatorname{len} \operatorname{maxPrefix}(p, q) < \operatorname{len} p$.

(9) For all finite sequences p, q such that $p \not\preceq q$ and $q \not\preceq p$ holds $p(\operatorname{len} \operatorname{maxPrefix}(p, q) + 1) \neq q(\operatorname{len} \operatorname{maxPrefix}(p, q) + 1)$.

2. Graph Preliminaries

Next we state three propositions:

(10) For every graph G and for every walk W of G and for all natural numbers m, n holds $\operatorname{len}(W.\operatorname{cut}(m, n)) \leq \operatorname{len} W$.

(11) Let G be a graph, W be a walk of G, and m, n be natural numbers. If $W.\operatorname{cut}(m, n)$ is non trivial, then W is non trivial.

(12) Let G be a graph, W be a walk of G, and m, n, i be odd natural numbers. Suppose $m \leq n \leq \operatorname{len} W$ and $i \leq \operatorname{len}(W.\operatorname{cut}(m, n))$. Then there exists an odd natural number j such that $(W.\operatorname{cut}(m, n))(i) = W(j)$ and $j = (m + i) - 1$ and $j \leq \operatorname{len} W$.

Let G be a graph. One can verify that every walk of G is non empty.

The following propositions are true:

(13) For every graph G and for all walks W_1, W_2 of G such that $W_1 \preceq W_2$ holds $W_1.\operatorname{vertices}() \subseteq W_2.\operatorname{vertices}()$.

(14) For every graph G and for all walks W_1, W_2 of G such that $W_1 \preceq W_2$ holds $W_1.\operatorname{edges}() \subseteq W_2.\operatorname{edges}()$.

(15) For every graph G and for all walks W_1, W_2 of G holds $W_1 \preceq W_1.\operatorname{append}(W_2)$.

(16) For every graph G and for all trails W_1, W_2 of G such that $W_1.\operatorname{last}() = W_2.\operatorname{first}()$ and $W_1.\operatorname{edges}()$ misses $W_2.\operatorname{edges}()$ holds $W_1.\operatorname{append}(W_2)$ is trail-like.

(17) Let G be a graph and P_1, P_2 be paths of G. Suppose $P_1.\operatorname{last}() = P_2.\operatorname{first}()$ and P_1 is open and P_2 is open and $P_1.\operatorname{edges}()$ misses $P_2.\operatorname{edges}()$ and if $P_1.\operatorname{first}() \in P_2.\operatorname{vertices}()$, then $P_1.\operatorname{first}() = P_2.\operatorname{last}()$ and $P_1.\operatorname{vertices}() \cap P_2.\operatorname{vertices}() \subseteq \{P_1.\operatorname{first}(), P_1.\operatorname{last}()\}$. Then $P_1.\operatorname{append}(P_2)$ is path-like.

(18) Let G be a graph and P_1, P_2 be paths of G. Suppose $P_1.\operatorname{last}() = P_2.\operatorname{first}()$ and P_1 is open and P_2 is open and $P_1.\operatorname{vertices}() \cap P_2.\operatorname{vertices}() = \{P_1.\operatorname{last}()\}$. Then $P_1.\operatorname{append}(P_2)$ is open and path-like.

(19) Let G be a graph and P_1, P_2 be paths of G. Suppose $P_1.\operatorname{last}() = P_2.\operatorname{first}()$ and $P_2.\operatorname{last}() = P_1.\operatorname{first}()$ and P_1 is open and P_2 is open and $P_1.\operatorname{edges}()$

misses P_2.edges() and P_1.vertices() $\cap$ P_2.vertices() $= \{P_1$.last(), P_1.first()$\}$. Then P_1.append(P_2) is cycle-like.

(20) Let G be a simple graph, W_1, W_2 be walks of G, and k be an odd natural number. Suppose $k \leq \operatorname{len} W_1$ and $k \leq \operatorname{len} W_2$ and for every odd natural number j such that $j \leq k$ holds $W_1(j) = W_2(j)$. Let j be a natural number. If $1 \leq j \leq k$, then $W_1(j) = W_2(j)$.

(21) For every graph G and for all walks W_1, W_2 of G such that W_1.first() $=$ W_2.first() holds len maxPrefix(W_1, W_2) is odd.

(22) For every graph G and for all walks W_1, W_2 of G such that W_1.first() $=$ W_2.first() and $W_1 \not\preceq W_2$ holds len maxPrefix$(W_1, W_2) + 2 \leq \operatorname{len} W_1$.

(23) For every non-multi graph G and for all walks W_1, W_2 of G such that W_1.first() $=$ W_2.first() and $W_1 \not\preceq W_2$ and $W_2 \not\preceq W_1$ holds W_1(len maxPrefix$(W_1, W_2) + 2) \neq W_2$(len maxPrefix$(W_1, W_2) + 2)$.

3. Trees

A tree is a tree-like graph. Let G be a graph. A subtree of G is a tree-like subgraph of G.

Let T be a tree. Observe that every walk of T which is trail-like is also path-like.

One can prove the following proposition

(24) For every tree T and for every path P of T such that P is non trivial holds P is open.

Let T be a tree. Note that every path of T which is non trivial is also open.

The following propositions are true:

(25) Let T be a tree, P be a path of T, and i, j be odd natural numbers. If $i < j \leq \operatorname{len} P$, then $P(i) \neq P(j)$.

(26) Let T be a tree, a, b be vertices of T, and P_1, P_2 be paths of T. If P_1 is walk from a to b and P_2 is walk from a to b, then $P_1 = P_2$.

Let T be a tree and let a, b be vertices of T. The functor T.pathBetween(a, b) yields a path of T and is defined as follows:

(Def. 2) T.pathBetween(a, b) is walk from a to b.

One can prove the following propositions:

(27) For every tree T and for all vertices a, b of T holds (T.pathBetween(a, b)).first() $= a$ and (T.pathBetween(a, b)).last() $= b$.

(28) For every tree T and for all vertices a, b of T holds $a, b \in$ (T.pathBetween(a, b)).vertices().

Let T be a tree and let a be a vertex of T. Observe that T.pathBetween(a, a) is closed.

Let T be a tree and let a be a vertex of T.

One can check that T.pathBetween(a, a) is trivial.

We now state a number of propositions:

(29) For every tree T and for every vertex a of T holds $(T.\mathrm{pathBetween}(a, a)).\mathrm{vertices}() = \{a\}$.

(30) For every tree T and for all vertices a, b of T holds $(T.\mathrm{pathBetween}(a, b)).\mathrm{reverse}() = T.\mathrm{pathBetween}(b, a)$.

(31) For every tree T and for all vertices a, b of T holds $(T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}() = (T.\mathrm{pathBetween}(b, a)).\mathrm{vertices}()$.

(32) Let T be a tree, a, b be vertices of T, t be a subtree of T, and a', b' be vertices of t. If $a = a'$ and $b = b'$, then $T.\mathrm{pathBetween}(a, b) = t.\mathrm{pathBetween}(a', b')$.

(33) Let T be a tree, a, b be vertices of T, and t be a subtree of T. Suppose $a \in$ the vertices of t and $b \in$ the vertices of t. Then $(T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}() \subseteq$ the vertices of t.

(34) Let T be a tree, P be a path of T, a, b be vertices of T, and i, j be odd natural numbers. If $i \leq j \leq \mathrm{len}\, P$ and $P(i) = a$ and $P(j) = b$, then $T.\mathrm{pathBetween}(a, b) = P.\mathrm{cut}(i, j)$.

(35) For every tree T and for all vertices a, b, c of T holds $c \in (T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}()$ iff $T.\mathrm{pathBetween}(a, b) = (T.\mathrm{pathBetween}(a, c)).\mathrm{append}((T.\mathrm{pathBetween}(c, b)))$.

(36) For every tree T and for all vertices a, b, c of T holds $c \in (T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}()$ iff $T.\mathrm{pathBetween}(a, c) \preceq T.\mathrm{pathBetween}(a, b)$.

(37) For every tree T and for all paths P_1, P_2 of T such that $P_1.\mathrm{last}() = P_2.\mathrm{first}()$ and $P_1.\mathrm{vertices}() \cap P_2.\mathrm{vertices}() = \{P_1.\mathrm{last}()\}$ holds $P_1.\mathrm{append}(P_2)$ is path-like.

(38) For every tree T and for all vertices a, b, c of T holds $c \in (T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}()$ iff $(T.\mathrm{pathBetween}(a, c)).\mathrm{vertices}() \cap (T.\mathrm{pathBetween}(c, b)).\mathrm{vertices}() = \{c\}$.

(39) Let T be a tree, a, b, c, d be vertices of T, and P_1, P_2 be paths of T. Suppose $P_1 = T.\mathrm{pathBetween}(a, b)$ and $P_2 = T.\mathrm{pathBetween}(a, c)$ and $P_1 \not\preceq P_2$ and $P_2 \not\preceq P_1$ and $d = P_1(\mathrm{len}\,\mathrm{maxPrefix}(P_1, P_2))$. Then $(T.\mathrm{pathBetween}(d, b)).\mathrm{vertices}() \cap (T.\mathrm{pathBetween}(d, c)).\mathrm{vertices}() = \{d\}$.

Let T be a tree and let a, b, c be vertices of T. The functor middleVertex(a, b, c) yielding a vertex of T is defined as follows:

(Def. 3) $(T.\mathrm{pathBetween}(a, b)).\mathrm{vertices}() \cap (T.\mathrm{pathBetween}(b, c)).\mathrm{vertices}() \cap (T.\mathrm{pathBetween}(c, a)).\mathrm{vertices}() = \{\mathrm{middleVertex}(a, b, c)\}$.

We now state a number of propositions:

(40) For every tree T and for all vertices a, b, c of T holds middleVertex$(a, b, c) =$ middleVertex(a, c, b).

(41) For every tree T and for all vertices a, b, c of T holds middleVertex$(a, b, c) =$ middleVertex(b, a, c).

(42) For every tree T and for all vertices a, b, c of T holds middleVertex$(a, b, c) =$ middleVertex(b, c, a).

(43) For every tree T and for all vertices a, b, c of T holds middleVertex$(a, b, c) =$ middleVertex(c, a, b).

(44) For every tree T and for all vertices a, b, c of T holds middleVertex$(a, b, c) =$ middleVertex(c, b, a).

(45) For every tree T and for all vertices a, b, c of T such that $c \in$ (T.pathBetween(a, b)).vertices() holds middleVertex$(a, b, c) = c$.

(46) For every tree T and for every vertex a of T holds middleVertex$(a, a, a) =$ a.

(47) For every tree T and for all vertices a, b of T holds middleVertex$(a, a, b) =$ a.

(48) For every tree T and for all vertices a, b of T holds middleVertex$(a, b, a) =$ a.

(49) For every tree T and for all vertices a, b of T holds middleVertex$(a, b, b) =$ b.

(50) Let T be a tree, P_1, P_2 be paths of T, and a, b, c be vertices of T. If $P_1 = T$.pathBetween(a, b) and $P_2 = T$.pathBetween(a, c) and $b \notin P_2$.vertices() and $c \notin P_1$.vertices(), then middleVertex$(a, b, c) =$ $P_1(\mathrm{len}\,\mathrm{maxPrefix}(P_1, P_2))$.

(51) Let T be a tree, P_1, P_2, P_3, P_4 be paths of T, and a, b, c be vertices of T. Suppose $P_1 = T$.pathBetween(a, b) and $P_2 = T$.pathBetween(a, c) and $P_3 = T$.pathBetween(b, a) and $P_4 = T$.pathBetween(b, c) and $b \notin P_2$.vertices() and $c \notin P_1$.vertices() and $a \notin P_4$.vertices(). Then $P_1(\mathrm{len}\,\mathrm{maxPrefix}(P_1, P_2)) = P_3(\mathrm{len}\,\mathrm{maxPrefix}(P_3, P_4))$.

(52) Let T be a tree, a, b, c be vertices of T, and S be a non empty set. Suppose that for every set s such that $s \in S$ holds there exists a subtree t of T such that $s =$ the vertices of t but a, $b \in s$ or a, $c \in s$ or b, $c \in s$. Then $\bigcap S \neq \emptyset$.

4. The Helly Property

Let F be a set. We say that F has Helly property if and only if:

(Def. 4) For every non empty set H such that $H \subseteq F$ and for all sets x, y such that x, $y \in H$ holds x meets y holds $\bigcap H \neq \emptyset$.

One can prove the following proposition

(53) Let T be a tree and X be a finite set such that for every set x such that $x \in X$ there exists a subtree t of T such that $x =$ the vertices of t. Then X has Helly property.

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Received January 10, 2008