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# Inner Products, Group, Ring of Quaternion Numbers

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**Summary.** In this article, we define the division of the quaternion numbers, we also give the definition of inner products, group, ring of the quaternion numbers, and we prove some of their properties.

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The articles [9], [1], [3], [4], [6], [5], [2], [7], and [8] provide the notation and terminology for this paper.

We use the following convention: $q$, $r$, $c$, $c_1$, $c_2$, $c_3$ are quaternion numbers and $x_1$, $x_2$, $x_3$, $x_4$, $y_1$, $y_2$, $y_3$, $y_4$ are elements of $\mathbb{R}$.

$0_{\mathbb{H}}$ is an element of $\mathbb{H}$.

$1_{\mathbb{H}}$ is an element of $\mathbb{H}$.

Next we state several propositions:

(1) For all real numbers $x$, $y$, $z$, $w$ holds $\langle x, y, z, w\rangle_{\mathbb{H}} = x + y \cdot i + z \cdot j + w \cdot k$.

(2) $(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$.

(3) $c + 0_{\mathbb{H}} = c$.

(4) $-\langle x_1, x_2, x_3, x_4\rangle_{\mathbb{H}} = \langle -x_1, -x_2, -x_3, -x_4\rangle_{\mathbb{H}}$.

(5) $\langle x_1, x_2, x_3, x_4\rangle_{\mathbb{H}} - \langle y_1, y_2, y_3, y_4\rangle_{\mathbb{H}} = \langle x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4\rangle_{\mathbb{H}}$.

(6) $(c_1 - c_2) + c_3 = (c_1 + c_3) - c_2$.

(7) $c_1 = (c_1 + c_2) - c_2$.

(8) $c_1 = (c_1 - c_2) + c_2$.

(9) $(-x_1) \cdot c = -x_1 \cdot c$.




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Let us consider $q$. Then $|q|$ is an element of $\mathbb{R}$.

$i$ Is an element of $\mathbb{H}$.

We now state a number of propositions:

(10) If $r \neq 0$, then $|r| > 0$.

(11) $(0) \cdot c = 0$.

(12) $c \cdot (0) = 0$.

(13) $c \cdot 1_{\mathbb{H}} = c$.

(14) $1_{\mathbb{H}} \cdot c = c$.

(15) $(c_1 \cdot c_2) \cdot c_3 = c_1 \cdot (c_2 \cdot c_3)$.

(16) $c_1 \cdot (c_2 + c_3) = c_1 \cdot c_2 + c_1 \cdot c_3$.

(17) $(c_1 + c_2) \cdot c_3 = c_1 \cdot c_3 + c_2 \cdot c_3$.

(18) $-c = (-1_{\mathbb{H}}) \cdot c$.

(19) $(-c_1) \cdot c_2 = -c_1 \cdot c_2$.

(20) $c_1 \cdot -c_2 = -c_1 \cdot c_2$.

(21) $(-c_1) \cdot -c_2 = c_1 \cdot c_2$.

(22) $(c_1 - c_2) \cdot c_3 = c_1 \cdot c_3 - c_2 \cdot c_3$.

(23) $c_1 \cdot (c_2 - c_3) = c_1 \cdot c_2 - c_1 \cdot c_3$.

(24) $\overline{\langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}} = \langle x_1, -x_2, -x_3, -x_4 \rangle_{\mathbb{H}}$.

(25) $\overline{\overline{c}} = c$.

Let us consider $q$, $r$. The functor $\frac{q}{r}$ is defined by the condition (Def. 1).

(Def. 1) There exist elements $q_0$, $q_1$, $q_2$, $q_3$, $r_0$, $r_1$, $r_2$, $r_3$ of $\mathbb{R}$ such that

(i) $q = \langle q_0, q_1, q_2, q_3 \rangle_{\mathbb{H}}$,

(ii) $r = \langle r_0, r_1, r_2, r_3 \rangle_{\mathbb{H}}$, and

(iii) $\frac{q}{r} = \langle \frac{r_0 \cdot q_0 + r_1 \cdot q_1 + r_2 \cdot q_2 + r_3 \cdot q_3}{|r|^{\mathbf{2}}}, \frac{(r_0 \cdot q_1 - r_1 \cdot q_0 - r_2 \cdot q_3) + r_3 \cdot q_2}{|r|^{\mathbf{2}}}, \frac{(r_0 \cdot q_2 + r_1 \cdot q_3) - r_2 \cdot q_0 - r_3 \cdot q_1}{|r|^{\mathbf{2}}}, \frac{((r_0 \cdot q_3 - r_1 \cdot q_2) + r_2 \cdot q_1) - r_3 \cdot q_0}{|r|^{\mathbf{2}}} \rangle_{\mathbb{H}}$.

Let us consider $q$, $r$. One can check that $\frac{q}{r}$ is quaternion.

Let us consider $q$, $r$. Then $\frac{q}{r}$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 2) $\frac{q}{r} = \frac{\Re(r) \cdot \Re(q) + \Im_1(q) \cdot \Im_1(r) + \Im_2(r) \cdot \Im_2(q) + \Im_3(r) \cdot \Im_3(q)}{|r|^{\mathbf{2}}} + \frac{(\Re(r) \cdot \Im_1(q) - \Im_1(r) \cdot \Re(q) - \Im_2(r) \cdot \Im_3(q)) + \Im_3(r) \cdot \Im_2(q)}{|r|^{\mathbf{2}}} \cdot i + \frac{(\Re(r) \cdot \Im_2(q) + \Im_1(r) \cdot \Im_3(q)) - \Im_2(r) \cdot \Re(q) - \Im_3(r) \cdot \Im_1(q)}{|r|^{\mathbf{2}}} \cdot j + \frac{((\Re(r) \cdot \Im_3(q) - \Im_1(r) \cdot \Im_2(q)) + \Im_2(r) \cdot \Im_1(q)) - \Im_3(r) \cdot \Re(q)}{|r|^{\mathbf{2}}} \cdot k$.

Let us consider $c$. The functor $c^{-1}$ yielding a quaternion number is defined by:

(Def. 3) $c^{-1} = \frac{1_{\mathbb{H}}}{c}$.

Let us consider $r$. Then $r^{-1}$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 4) $r^{-1} = \frac{\Re(r)}{|r|^{\mathbf{2}}} - \frac{\Im_1(r)}{|r|^{\mathbf{2}}} \cdot i - \frac{\Im_2(r)}{|r|^{\mathbf{2}}} \cdot j - \frac{\Im_3(r)}{|r|^{\mathbf{2}}} \cdot k$.

We now state several propositions:

(26) $\Re(r^{-1}) = \frac{\Re(r)}{|r|^{\mathbf{2}}}$ and $\Im_1(r^{-1}) = -\frac{\Im_1(r)}{|r|^{\mathbf{2}}}$ and $\Im_2(r^{-1}) = -\frac{\Im_2(r)}{|r|^{\mathbf{2}}}$ and $\Im_3(r^{-1}) = -\frac{\Im_3(r)}{|r|^{\mathbf{2}}}$.

(27)(i) $\Re(\frac{q}{r}) = \frac{\Re(r)\cdot\Re(q)+\Im_1(q)\cdot\Im_1(r)+\Im_2(r)\cdot\Im_2(q)+\Im_3(r)\cdot\Im_3(q)}{|r|^{\mathbf{2}}}$,

(ii) $\Im_1(\frac{q}{r}) = \frac{(\Re(r)\cdot\Im_1(q)-\Im_1(r)\cdot\Re(q)-\Im_2(r)\cdot\Im_3(q))+\Im_3(r)\cdot\Im_2(q)}{|r|^{\mathbf{2}}}$,

(iii) $\Im_2(\frac{q}{r}) = \frac{(\Re(r)\cdot\Im_2(q)+\Im_1(r)\cdot\Im_3(q))-\Im_2(r)\cdot\Re(q)-\Im_3(r)\cdot\Im_1(q)}{|r|^{\mathbf{2}}}$, and

(iv) $\Im_3(\frac{q}{r}) = \frac{((\Re(r)\cdot\Im_3(q)-\Im_1(r)\cdot\Im_2(q))+\Im_2(r)\cdot\Im_1(q))-\Im_3(r)\cdot\Re(q)}{|r|^{\mathbf{2}}}$.

(28) If $r \neq 0$, then $r \cdot r^{-1} = 1$.

(29) If $r \neq 0$, then $r^{-1} \cdot r = 1$.

(30) If $c \neq 0_{\mathbb{H}}$, then $\frac{c}{c} = 1_{\mathbb{H}}$.

(31) $(-c)^{-1} = -c^{-1}$.

The unary operation $\mathrm{compl}_{\mathbb{H}}$ on $\mathbb{H}$ is defined by:

(Def. 5) For every element $c$ of $\mathbb{H}$ holds $\mathrm{compl}_{\mathbb{H}}(c) = -c$.

The binary operation $+_{\mathbb{H}}$ on $\mathbb{H}$ is defined as follows:

(Def. 6) For all elements $c_1$, $c_2$ of $\mathbb{H}$ holds $+_{\mathbb{H}}(c_1, c_2) = c_1 + c_2$.

The binary operation $-_{\mathbb{H}}$ on $\mathbb{H}$ is defined by:

(Def. 7) For all elements $c_1$, $c_2$ of $\mathbb{H}$ holds $-_{\mathbb{H}}(c_1, c_2) = c_1 - c_2$.

The binary operation $\cdot_{\mathbb{H}}$ on $\mathbb{H}$ is defined as follows:

(Def. 8) For all elements $c_1$, $c_2$ of $\mathbb{H}$ holds $\cdot_{\mathbb{H}}(c_1, c_2) = c_1 \cdot c_2$.

The binary operation $/_{\mathbb{H}}$ on $\mathbb{H}$ is defined as follows:

(Def. 9) For all elements $c_1$, $c_2$ of $\mathbb{H}$ holds $/_{\mathbb{H}}(c_1, c_2) = \frac{c_1}{c_2}$.

The unary operation ${}_{\mathbb{H}}^{-1}$ on $\mathbb{H}$ is defined by:

(Def. 10) For every element $c$ of $\mathbb{H}$ holds $({}_{\mathbb{H}}^{-1})(c) = c^{-1}$.

The strict additive loop structure $\mathbb{H}_{\mathrm{G}}$ is defined as follows:

(Def. 11) The carrier of $\mathbb{H}_{\mathrm{G}} = \mathbb{H}$ and the addition of $\mathbb{H}_{\mathrm{G}} = +_{\mathbb{H}}$ and $0_{\mathbb{H}_{\mathrm{G}}} = 0_{\mathbb{H}}$.

Let us mention that $\mathbb{H}_{\mathrm{G}}$ is non empty.

Let us note that every element of $\mathbb{H}_{\mathrm{G}}$ is quaternion.

Let $x$, $y$ be elements of $\mathbb{H}_{\mathrm{G}}$ and let $a$, $b$ be quaternion numbers. One can check that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

One can prove the following proposition

(32) $0_{\mathbb{H}_{\mathrm{G}}} = 0_{\mathbb{H}}$.

Let us observe that $\mathbb{H}_{\mathrm{G}}$ is Abelian, add-associative, right zeroed, and right complementable.

Let $x$ be an element of $\mathbb{H}_{\mathrm{G}}$ and let $a$ be a quaternion number. Note that $-x$ and $-a$ can be identified when $x = a$.

Let $x$, $y$ be elements of $\mathbb{H}_{\rm G}$ and let $a$, $b$ be quaternion numbers. One can verify that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

Next we state the proposition

(33) For all elements $x$, $y$, $z$ of $\mathbb{H}_{\rm G}$ holds $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathbb{H}_{\rm G}} = x$.

The strict double loop structure $\mathbb{H}_{\rm R}$ is defined as follows:

(Def. 12) The carrier of $\mathbb{H}_{\rm R} = \mathbb{H}$ and the addition of $\mathbb{H}_{\rm R} = +_{\mathbb{H}}$ and the multiplication of $\mathbb{H}_{\rm R} = \cdot_{\mathbb{H}}$ and $1_{\mathbb{H}_{\rm R}} = 1_{\mathbb{H}}$ and $0_{\mathbb{H}_{\rm R}} = 0_{\mathbb{H}}$.

Let us note that $\mathbb{H}_{\rm R}$ is non empty.

Let us observe that every element of $\mathbb{H}_{\rm R}$ is quaternion.

Let $a$, $b$ be quaternion numbers and let $x$, $y$ be elements of $\mathbb{H}_{\rm R}$. One can check the following observations: $x + y$ can be identified with $a + b$ and $x \cdot y$ can be identified with $a \cdot b$ when $x = a$ and $y = b$.

One can check that $\mathbb{H}_{\rm R}$ is well unital.

Next we state three propositions:

(34) $1_{\mathbb{H}_{\rm R}} = 1_{\mathbb{H}}$.

(35) $\mathbf{1}_{\mathbb{H}_{\rm R}} = 1_{\mathbb{H}}$.

(36) $0_{\mathbb{H}_{\rm R}} = 0_{\mathbb{H}}$.

Let us mention that $\mathbb{H}_{\rm R}$ is add-associative, right zeroed, right complementable, Abelian, associative, left unital, right unital, distributive, almost right invertible, and non degenerated.

Let $x$ be an element of $\mathbb{H}_{\rm R}$ and let $a$ be a quaternion number. Observe that $-x$ and $-a$ can be identified when $x = a$.

Let $x$, $y$ be elements of $\mathbb{H}_{\rm R}$ and let $a$, $b$ be quaternion numbers. Observe that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

Let $z$ be an element of $\mathbb{H}_{\rm R}$. Then $\overline{z}$ is an element of $\mathbb{H}_{\rm R}$.

In the sequel $z$ denotes an element of $\mathbb{H}_{\rm R}$.

The following propositions are true:

(37) $-z = (-\mathbf{1}_{\mathbb{H}_{\rm R}}) \cdot z$.

(38) $\overline{0_{\mathbb{H}_{\rm R}}} = 0_{\mathbb{H}_{\rm R}}$.

(39) If $\overline{z} = 0_{\mathbb{H}_{\rm R}}$, then $z = 0_{\mathbb{H}_{\rm R}}$.

(40) $\overline{1_{\mathbb{H}_{\rm R}}} = 1_{\mathbb{H}_{\rm R}}$.

(41) $|0_{\mathbb{H}_{\rm R}}| = 0$.

(42) If $|z| = 0$, then $z = 0_{\mathbb{H}_{\rm R}}$.

(43) $|1_{\mathbb{H}_{\rm R}}| = 1$.

(44) $(1_{\mathbb{H}_{\rm R}})^{-1} = 1_{\mathbb{H}_{\rm R}}$.

Let $x$, $y$ be quaternion numbers. The functor $(x|y)$ yielding an element of $\mathbb{H}$ is defined as follows:

(Def. 13) $(x|y) = x \cdot \overline{y}$.

The following propositions are true:

(45) $(c_1|c_2) = \langle \Re(c_1) \cdot \Re(c_2) + \Im_1(c_1) \cdot \Im_1(c_2) + \Im_2(c_1) \cdot \Im_2(c_2) + \Im_3(c_1) \cdot \Im_3(c_2), ((\Re(c_1) \cdot -\Im_1(c_2) + \Im_1(c_1) \cdot \Re(c_2)) - \Im_2(c_1) \cdot \Im_3(c_2)) + \Im_3(c_1) \cdot \Im_2(c_2), ((\Re(c_1) \cdot -\Im_2(c_2) + \Re(c_2) \cdot \Im_2(c_1)) - \Im_1(c_2) \cdot \Im_3(c_1)) + \Im_3(c_2) \cdot \Im_1(c_1), ((\Re(c_1) \cdot -\Im_3(c_2) + \Im_3(c_1) \cdot \Re(c_2)) - \Im_1(c_1) \cdot \Im_2(c_2)) + \Im_2(c_1) \cdot \Im_1(c_2) \rangle_{\mathbb{H}}$.

(46) $(c|c) = |c|^{\mathbf{2}}$.

(47) $\Re((c|c)) = |c|^{\mathbf{2}}$ and $\Im_1((c|c)) = 0$ and $\Im_2((c|c)) = 0$ and $\Im_2((c|c)) = 0$.

(48) $|(c_1|c_2)| = |c_1| \cdot |c_2|$.

(49) If $(c|c) = 0$, then $c = 0$.

(50) $((c_1 + c_2)|c_3) = (c_1|c_3) + (c_2|c_3)$.

(51) $(c_1|(c_2 + c_3)) = (c_1|c_2) + (c_1|c_3)$.

(52) $((-c_1)|c_2) = -(c_1|c_2)$.

(53) $-(c_1|c_2) = (c_1|-c_2)$.

(54) $((-c_1)|-c_2) = (c_1|c_2)$.

(55) $((c_1 - c_2)|c_3) = (c_1|c_3) - (c_2|c_3)$.

(56) $(c_1|(c_2 - c_3)) = (c_1|c_2) - (c_1|c_3)$.

(57) $((c_1 + c_2)|(c_1 + c_2)) = (c_1|c_1) + (c_1|c_2) + (c_2|c_1) + (c_2|c_2)$.

(58) $((c_1 - c_2)|(c_1 - c_2)) = ((c_1|c_1) - (c_1|c_2) - (c_2|c_1)) + (c_2|c_2)$.

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