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Uniqueness of Factoring an Integer and Multiplicative Group $\mathbb{Z}/p\mathbb{Z}^*$

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Summary. In the [20], it had been proven that the Integers modulo p, in this article we shall refer as $\mathbb{Z}/p\mathbb{Z}$, constitutes a field if and only if p is a prime. Then the prime modulo $\mathbb{Z}/p\mathbb{Z}$ is an additive cyclic group and $\mathbb{Z}/p\mathbb{Z}^* = \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ is a multiplicative cyclic group, too. The former has been proven in the [23]. However, the latter had not been proven yet. In this article, first, we prove a theorem concerning the LCM to prove the existence of primitive elements of $\mathbb{Z}/p^*$. Moreover we prove the uniqueness of factoring an integer. Next we define the multiplicative group $\mathbb{Z}/p\mathbb{Z}^*$ and prove it is cyclic.

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The articles [31], [3], [9], [1], [25], [2], [32], [8], [24], [4], [19], [29], [28], [13], [7], [26], [22], [11], [17], [18], [12], [16], [30], [23], [27], [5], [14], [15], [20], [21], [6], and [10] provide the terminology and notation for this paper.

1. Uniqueness of Factoring an Integer

In this paper x, X denote sets.

Next we state four propositions:

(1) For every many sorted set p indexed by X such that $\operatorname{support} p = \{x\}$ holds $p = (X \longmapsto 0) +\cdot (x, p(x))$.

(2) Let X be a set and p, q, r be real-valued many sorted sets indexed by X. If $\operatorname{support} p \cap \operatorname{support} q = \emptyset$ and $\operatorname{support} p \cup \operatorname{support} q = \operatorname{support} r$ and $p \restriction \operatorname{support} p = r \restriction \operatorname{support} p$ and $q \restriction \operatorname{support} q = r \restriction \operatorname{support} q$, then $p + q = r$.

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(3) For every set X and for all many sorted sets p, q indexed by X such that $p \restriction \operatorname{support} p = q \restriction \operatorname{support} q$ holds $p = q$.

(4) For every set X and for all bags p, q of X such that $\operatorname{support} p = \emptyset$ and $\operatorname{support} q = \emptyset$ holds $p = q$.

Let p be a bag of Prime. We say that p is prime-factorization-like if and only if:

(Def. 1) For every prime number x such that $x \in \operatorname{support} p$ there exists a natural number n such that $0 < n$ and $p(x) = x^n$.

Let n be a non empty natural number. Note that $\mathrm{PPF}(n)$ is prime-factorization-like.

Next we state a number of propositions:

(5) For all prime numbers p, q and for all natural numbers n, m such that $p \mid m \cdot q^n$ and $p \neq q$ holds $p \mid m$.

(6) Let f be a finite sequence of elements of $\mathbb{N}$, b be a bag of Prime, and a be a prime number. Suppose b is prime-factorization-like and $\prod b \neq 1$ and $a \mid \prod b$ and $\prod b = \prod f$ and $f = b \cdot \mathrm{CFS}(\operatorname{support} b)$. Then $a \in \operatorname{support} b$.

(7) For all bags p, q of Prime such that $\operatorname{support} p \subseteq \operatorname{support} q$ and $p \restriction \operatorname{support} p = q \restriction \operatorname{support} p$ holds $\prod p \mid \prod q$.

(8) Let p be a bag of Prime and x be a prime number. If p is prime-factorization-like, then $x \mid \prod p$ iff $x \in \operatorname{support} p$.

(9) For all non empty natural numbers n, m, k such that $k = \operatorname{lcm}(n, m)$ holds $\operatorname{support} \mathrm{PPF}(k) = \operatorname{support} \mathrm{PPF}(n) \cup \operatorname{support} \mathrm{PPF}(m)$.

(10) For every set X and for all bags b_1, b_2 of X holds $\operatorname{support} \min(b_1, b_2) = \operatorname{support} b_1 \cap \operatorname{support} b_2$.

(11) For all non empty natural numbers n, m, k such that $k = n \gcd m$ holds $\operatorname{support} \mathrm{PPF}(k) = \operatorname{support} \mathrm{PPF}(n) \cap \operatorname{support} \mathrm{PPF}(m)$.

(12) Let p, q be bags of Prime. Suppose p is prime-factorization-like and q is prime-factorization-like and $\operatorname{support} p$ misses $\operatorname{support} q$. Then $\prod p$ and $\prod q$ are relative prime.

(13) For every bag p of Prime such that p is prime-factorization-like holds $\prod p \neq 0$.

(14) For every bag p of Prime such that p is prime-factorization-like holds $\prod p = 1$ iff $\operatorname{support} p = \emptyset$.

(15) Let p, q be bags of Prime. Suppose p is prime-factorization-like and q is prime-factorization-like and $\prod p = \prod q$. Then $p = q$.

(16) Let p be a bag of Prime and n be a non empty natural number. If p is prime-factorization-like and $n = \prod p$, then $\mathrm{PPF}(n) = p$.

(17) Let n, m be elements of $\mathbb{N}$. Suppose $1 \leq n$ and $1 \leq m$. Then there exist elements m_0, n_0 of $\mathbb{N}$ such that $\operatorname{lcm}(n, m) = n_0 \cdot m_0$ and $n_0 \gcd m_0 = 1$

and $n_0 \mid n$ and $m_0 \mid m$ and $n_0 \neq 0$ and $m_0 \neq 0$.

2. Multiplicative Group $\mathbb{Z}/p\mathbb{Z}^*$

Let n be a natural number. Let us assume that $1 < n$. The functor $\mathbb{Z}_n^*$ yields a non empty finite subset of $\mathbb{N}$ and is defined by:

(Def. 2) $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$.

We now state the proposition

(18) For every natural number n such that $1 < n$ holds $\overline{\overline{\mathbb{Z}_n^*}} = n - 1$.

Let n be a prime number. The functor $\cdot_{\mathbb{Z}_n^*}$ yielding a binary operation on $\mathbb{Z}_n^*$ is defined by:

(Def. 3) $\cdot_{\mathbb{Z}_n^*} = \cdot_{\mathbb{Z}_n} \restriction \mathbb{Z}_n^*$.

One can prove the following proposition

(19) For every prime number p holds $\langle \mathbb{Z}_p^*, \cdot_{\mathbb{Z}_p^*} \rangle$ is associative, commutative, and group-like.

Let p be a prime number. The functor $\mathbb{Z}/p\mathbb{Z}^*$ yielding a commutative group is defined by:

(Def. 4) $\mathbb{Z}/p\mathbb{Z}^* = \langle \mathbb{Z}_p^*, \cdot_{\mathbb{Z}_p^*} \rangle$.

The following three propositions are true:

(20) Let p be a prime number, x, y be elements of $\mathbb{Z}/p\mathbb{Z}^*$, and x_1, y_1 be elements of $\mathbb{Z}_p^{\mathrm{R}}$. If $x = x_1$ and $y = y_1$, then $x \cdot y = x_1 \cdot y_1$.

(21) For every prime number p holds $\mathbf{1}_{\mathbb{Z}/p\mathbb{Z}^*} = 1$ and $\mathbf{1}_{\mathbb{Z}/p\mathbb{Z}^*} = 1_{\mathbb{Z}_p^{\mathrm{R}}}$.

(22) For every prime number p and for every element x of $\mathbb{Z}/p\mathbb{Z}^*$ and for every element x_1 of $\mathbb{Z}_p^{\mathrm{R}}$ such that $x = x_1$ holds $x^{-1} = {x_1}^{-1}$.

Let p be a prime number. One can verify that $\mathbb{Z}/p\mathbb{Z}^*$ is finite.

We now state several propositions:

(23) For every prime number p holds $\mathrm{ord}(\mathbb{Z}/p\mathbb{Z}^*) = p - 1$.

(24) Let G be a group, a be an element of G, and i be an integer. Suppose a is not of order 0. Then there exist elements n, k of $\mathbb{N}$ such that $a^i = a^n$ and $n = k \cdot \mathrm{ord}(a) + i$.

(25) Let G be a commutative group, a, b be elements of G, and n, m be natural numbers. If G is finite and $\mathrm{ord}(a) = n$ and $\mathrm{ord}(b) = m$ and $n \gcd m = 1$, then $\mathrm{ord}(a \cdot b) = n \cdot m$.

(26) For every non empty zero structure L and for every polynomial p of L such that $0 \leq \deg p$ holds p is non-zero.

(27) For every field L and for every polynomial f of L such that $0 \leq \deg f$ holds $\mathrm{Roots}\, f$ is a finite set and $\overline{\overline{\mathrm{Roots}\, f}} \leq \deg f$.

(28) Let p be a prime number, z be an element of $\mathbb{Z}/p\mathbb{Z}^*$, and y be an element of $\mathbb{Z}_p^{\mathrm{R}}$. If $z = y$, then for every element n of $\mathbb{N}$ holds $\mathrm{power}_{\mathbb{Z}/p\mathbb{Z}^*}(z, n) = \mathrm{power}_{\mathbb{Z}_p^{\mathrm{R}}}(y, n)$.

(29) Let p be a prime number, a, b be elements of $\mathbb{Z}/p\mathbb{Z}^*$, and n be a natural number. If $0 < n$ and $\mathrm{ord}(a) = n$ and $b^n = 1$, then b is an element of $\mathrm{gr}(\{a\})$.

(30) Let G be a group, z be an element of G, and d, l be elements of $\mathbb{N}$. If G is finite and $\mathrm{ord}(z) = d \cdot l$, then $\mathrm{ord}(z^d) = l$.

(31) For every prime number p holds $\mathbb{Z}/p\mathbb{Z}^*$ is a cyclic group.

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