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Laplace Expansion

Karol Pąk
Institute of Computer Science
University of Białystok
Poland

Andrzej Trybulec
Institute of Computer Science
University of Białystok
Poland

Summary. In the article the formula for Laplace expansion is proved.

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The notation and terminology used in this paper are introduced in the following articles: [23], [11], [29], [20], [12], [30], [31], [6], [9], [7], [3], [4], [21], [28], [26], [15], [22], [10], [5], [13], [24], [14], [33], [25], [18], [34], [1], [8], [2], [16], [17], [27], [19], and [32].

1. Preliminaries

For simplicity, we follow the rules: x, y are sets, N is an element of $\mathbb{N}$, c, i, j, k, m, n are natural numbers, D is a non empty set, s is an element of $2\mathrm{Set}\,\mathrm{Seg}(n+2)$, p is an element of the permutations of n-element set, p_1, q_1 are elements of the permutations of $(n+1)$-element set, p_2 is an element of the permutations of $(n+2)$-element set, K is a field, a, b are elements of K, f is a finite sequence of elements of K, A is a matrix over K, A_1 is a matrix over D of dimension $n \times m$, p_3 is a finite sequence of elements of D, and M is a matrix over K of dimension n.

The following propositions are true:

(1) For every finite sequence f and for every natural number i such that $i \in \mathrm{dom}\, f$ holds $\mathrm{len}(f_{\restriction i}) = \mathrm{len}\, f -' 1$.

(2) Let i, j, n be natural numbers and M be a matrix over K of dimension n. If $i \in \mathrm{dom}\, M$, then $\mathrm{len}\,$(the deleting of i-row and j-column in M) $= n -' 1$.

(3) If $j \in \mathrm{Seg}\,\mathrm{width}\, A$, then width (the deleting of j-column in A) $= \mathrm{width}\, A -' 1$.

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(4) For every natural number i such that $\operatorname{len} A > 1$ holds $\operatorname{width} A =$ width (the deleting of i-row in A).

(5) For every natural number i such that $j \in \operatorname{Seg} \operatorname{width} M$ holds width (the deleting of i-row and j-column in M) $= n -' 1$.

Let G be a non empty groupoid, let B be a function from $[\!:$ the carrier of G, $\mathbb{N} :\!]$ into the carrier of G, let g be an element of G, and let i be a natural number. Then $B(g, i)$ is an element of G.

One can prove the following propositions:

(6) $\overline{\overline{\text{the permutations of } n\text{-element set}}} = n!$.

(7) For all i, j such that $i \in \operatorname{Seg}(n+1)$ and $j \in \operatorname{Seg}(n+1)$ holds $\overline{\overline{\{p_1 : p_1(i) = j\}}} = n!$.

(8) Let K be a Fanoian field, given p_2, and X, Y be elements of $\operatorname{Fin} 2\operatorname{Set} \operatorname{Seg}(n+2)$. Suppose $Y = \{s : s \in X \wedge (\text{Part-sgn}(p_2, K))(s) = -\mathbf{1}_K\}$. Then (the multiplication of K)-$\sum_X \text{Part-sgn}(p_2, K) = \operatorname{power}_K(-\mathbf{1}_K, \operatorname{card} Y)$.

(9) Let K be a Fanoian field and given p_2, i, j. Suppose $i \in \operatorname{Seg}(n+2)$ and $p_2(i) = j$. Then there exists an element X of $\operatorname{Fin} 2\operatorname{Set} \operatorname{Seg}(n+2)$ such that $X = \{\{N, i\} : \{N, i\} \in 2\operatorname{Set} \operatorname{Seg}(n+2)\}$ and (the multiplication of K)-$\sum_X \text{Part-sgn}(p_2, K) = \operatorname{power}_K(-\mathbf{1}_K, i + j)$.

(10) Let given i, j. Suppose $i \in \operatorname{Seg}(n+1)$ and $j \in \operatorname{Seg}(n+1)$ and $n \geq 2$. Then there exists a function P_1 from $2\operatorname{Set} \operatorname{Seg} n$ into $2\operatorname{Set} \operatorname{Seg}(n+1)$ such that

(i) $\operatorname{rng} P_1 = 2\operatorname{Set} \operatorname{Seg}(n+1) \setminus \{\{N, i\} : \{N, i\} \in 2\operatorname{Set} \operatorname{Seg}(n+1)\}$,

(ii) P_1 is one-to-one, and

(iii) for all k, m such that $k < m$ and $\{k, m\} \in 2\operatorname{Set} \operatorname{Seg} n$ holds if $m < i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m\}$ and if $m \geq i$ and $k < i$, then $P_1(\{k, m\}) = \{k, m+1\}$ and if $m \geq i$ and $k \geq i$, then $P_1(\{k, m\}) = \{k+1, m+1\}$.

(11) If $n < 2$, then for every element p of the permutations of n-element set holds p is even and $p = \operatorname{idseq}(n)$.

(12) Let X, Y, D be non empty sets, f be a function from X into $\operatorname{Fin} Y$, g be a function from $\operatorname{Fin} Y$ into D, and F be a binary operation on D. Suppose that

(i) for all elements A, B of $\operatorname{Fin} Y$ such that A misses B holds $F(g(A), g(B)) = g(A \cup B)$,

(ii) F is commutative and associative and has a unity, and

(iii) $g(\emptyset) = \mathbf{1}_F$.

Let I be an element of $\operatorname{Fin} X$. Suppose that for all x, y such that $x \in I$ and $y \in I$ and $f(x)$ meets $f(y)$ holds $x = y$. Then F-$\sum_I g \cdot f = F$-$\sum_{f^\circ I} g$ and F-$\sum_{f^\circ I} g = g(\bigcup(f^\circ I))$ and $\bigcup(f^\circ I)$ is an element of $\operatorname{Fin} Y$.

2. Auxiliary Notions

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. Let us assume that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. The functor $\operatorname{Delete}(M, i, j)$ yielding a matrix over K of dimension $n -' 1$ is defined as follows:

(Def. 1) $\operatorname{Delete}(M, i, j) =$ the deleting of i-row and j-column in M.

The following propositions are true:

(13) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Let given k, m such that $k \in \operatorname{Seg}(n -' 1)$ and $m \in \operatorname{Seg}(n -' 1)$. Then

(i) if $k < i$ and $m < j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k,m}$,

(ii) if $k < i$ and $m \geq j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k,m+1}$,

(iii) if $k \geq i$ and $m < j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k+1,m}$, and

(iv) if $k \geq i$ and $m \geq j$, then $(\operatorname{Delete}(M, i, j))_{k,m} = M_{k+1,m+1}$.

(14) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ holds $(\operatorname{Delete}(M, i, j))^{\mathrm{T}} = \operatorname{Delete}(M^{\mathrm{T}}, j, i)$.

(15) For every finite sequence f of elements of K and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(M, i, j) = \operatorname{Delete}(\operatorname{RLine}(M, i, f), i, j)$.

Let us consider c, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D. The functor $\operatorname{ReplaceCol}(M, c, p_3)$ yielding a matrix over D of dimension $n \times m$ is defined by:

(Def. 2)(i) $\operatorname{len} \operatorname{ReplaceCol}(M, c, p_3) = \operatorname{len} M$ and $\operatorname{width} \operatorname{ReplaceCol}(M, c, p_3) = \operatorname{width} M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $j \neq c$, then $(\operatorname{ReplaceCol}(M, c, p_3))_{i,j} = M_{i,j}$ and if $j = c$, then $(\operatorname{ReplaceCol}(M, c, p_3))_{i,c} = p_3(i)$ if $\operatorname{len} p_3 = \operatorname{len} M$,

(ii) $\operatorname{ReplaceCol}(M, c, p_3) = M$, otherwise.

Let us consider c, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_3 be a finite sequence of elements of D. We introduce $\operatorname{RCol}(M, c, p_3)$ as a synonym of $\operatorname{ReplaceCol}(M, c, p_3)$.

We now state four propositions:

(16) For every i such that $i \in \operatorname{Seg} \operatorname{width} A_1$ holds if $i = c$ and $\operatorname{len} p_3 = \operatorname{len} A_1$, then $(\operatorname{RCol}(A_1, c, p_3))_{\square,i} = p_3$ and if $i \neq c$, then $(\operatorname{RCol}(A_1, c, p_3))_{\square,i} = (A_1)_{\square,i}$.

(17) If $c \notin \operatorname{Seg} \operatorname{width} A_1$, then $\operatorname{RCol}(A_1, c, p_3) = A_1$.

(18) $\operatorname{RCol}(A_1, c, (A_1)_{\square,c}) = A_1$.

(19) Let A be a matrix over D of dimension $n \times m$ and A' be a matrix over D of dimension $m \times n$. If $A' = A^{\mathrm{T}}$ and if $m = 0$, then $n = 0$, then $\operatorname{ReplaceCol}(A, c, p_3) = (\operatorname{ReplaceLine}(A', c, p_3))^{\mathrm{T}}$.

3. Permutations

Let us consider i, n and let p_4 be an element of the permutations of $(n+1)$-element set. Let us assume that $i \in \mathrm{Seg}(n+1)$. The functor $\mathrm{Rem}(p_4, i)$ yielding an element of the permutations of n-element set is defined by the condition (Def. 3).

(Def. 3) Let given k such that $k \in \mathrm{Seg}\, n$. Then

(i) if $k < i$, then if $p_4(k) < p_4(i)$, then $(\mathrm{Rem}(p_4, i))(k) = p_4(k)$ and if $p_4(k) \geq p_4(i)$, then $(\mathrm{Rem}(p_4, i))(k) = p_4(k) - 1$, and

(ii) if $k \geq i$, then if $p_4(k+1) < p_4(i)$, then $(\mathrm{Rem}(p_4, i))(k) = p_4(k+1)$ and if $p_4(k+1) \geq p_4(i)$, then $(\mathrm{Rem}(p_4, i))(k) = p_4(k+1) - 1$.

One can prove the following three propositions:

(20) Let given i, j. Suppose $i \in \mathrm{Seg}(n+1)$ and $j \in \mathrm{Seg}(n+1)$. Let P be a set. Suppose $P = \{p_1 : p_1(i) = j\}$. Then there exists a function P_1 from P into the permutations of n-element set such that P_1 is bijective and for every q_1 such that $q_1(i) = j$ holds $P_1(q_1) = \mathrm{Rem}(q_1, i)$.

(21) For all i, j such that $i \in \mathrm{Seg}(n+1)$ and $p_1(i) = j$ holds $(-1)^{\mathrm{sgn}(p_1)} a = \mathrm{power}_K(-\mathbf{1}_K, i+j) \cdot (-1)^{\mathrm{sgn}(\mathrm{Rem}(p_1, i))} a$.

(22) Let given i, j. Suppose $i \in \mathrm{Seg}(n+1)$ and $p_1(i) = j$. Let M be a matrix over K of dimension $n+1$ and D_1 be a matrix over K of dimension n. Suppose $D_1 = \mathrm{Delete}(M, i, j)$. Then (the product on paths of M)$(p_1) = \mathrm{power}_K(-\mathbf{1}_K, i+j) \cdot M_{i,j} \cdot$ (the product on paths of D_1)$(\mathrm{Rem}(p_1, i))$.

4. Minors and Cofactors

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{Minor}(M, i, j)$ yielding an element of K is defined by:

(Def. 4) $\mathrm{Minor}(M, i, j) = \mathrm{Det}\,\mathrm{Delete}(M, i, j)$.

Let i, j, n be natural numbers, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{Cofactor}(M, i, j)$ yielding an element of K is defined as follows:

(Def. 5) $\mathrm{Cofactor}(M, i, j) = \mathrm{power}_K(-\mathbf{1}_K, i+j) \cdot \mathrm{Minor}(M, i, j)$.

The following propositions are true:

(23) Let given i, j. Suppose $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$. Let P be an element of Fin (the permutations of n-element set). Suppose $P = \{p : p(i) = j\}$. Let M be a matrix over K of dimension n. Then (the addition of K)-$\sum_P$ (the product on paths of M) $= M_{i,j} \cdot \mathrm{Cofactor}(M, i, j)$.

(24) For all i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ holds $\mathrm{Minor}(M, i, j) = \mathrm{Minor}(M^{\mathrm{T}}, j, i)$.

Let us consider n, K and let M be a matrix over K of dimension n. The matrix of cofactor M yielding a matrix over K of dimension n is defined by the condition (Def. 6).

(Def. 6) Let i, j be natural numbers. Suppose $\langle i, j\rangle \in$ the indices of the matrix of cofactor M. Then (the matrix of cofactor $M)_{i,j} = \mathrm{Cofactor}(M, i, j)$.

5. Laplace Expansion

Let us consider n, i, K and let M be a matrix over K of dimension n. The functor $\mathrm{LaplaceExpL}(M, i)$ yields a finite sequence of elements of K and is defined as follows:

(Def. 7) $\mathrm{len}\,\mathrm{LaplaceExpL}(M, i) = n$ and for every j such that $j \in \mathrm{dom}\,\mathrm{LaplaceExpL}(M, i)$ holds $(\mathrm{LaplaceExpL}(M, i))(j) = M_{i,j} \cdot \mathrm{Cofactor}(M, i, j)$.

Let us consider n, let j be a natural number, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{LaplaceExpC}(M, j)$ yields a finite sequence of elements of K and is defined by:

(Def. 8) $\mathrm{len}\,\mathrm{LaplaceExpC}(M, j) = n$ and for every i such that $i \in \mathrm{dom}\,\mathrm{LaplaceExpC}(M, j)$ holds $(\mathrm{LaplaceExpC}(M, j))(i) = M_{i,j} \cdot \mathrm{Cofactor}(M, i, j)$.

One can prove the following propositions:

(25) For every natural number i and for every matrix M over K of dimension n such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{Det}\,M = \sum \mathrm{LaplaceExpL}(M, i)$.

(26) For every i such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{LaplaceExpC}(M, i) = \mathrm{LaplaceExpL}(M^{\mathrm{T}}, i)$.

(27) For every natural number j and for every matrix M over K of dimension n such that $j \in \mathrm{Seg}\,n$ holds $\mathrm{Det}\,M = \sum \mathrm{LaplaceExpC}(M, j)$.

(28) For all p, i such that $\mathrm{len}\,f = n$ and $i \in \mathrm{Seg}\,n$ holds Line(the matrix of cofactor M, $i) \bullet f = \mathrm{LaplaceExpL}(\mathrm{RLine}(M, i, f), i)$.

(29) If $i \in \mathrm{Seg}\,n$, then $\mathrm{Line}(M, j) \cdot (($the matrix of cofactor $M)^{\mathrm{T}})_{\square,i} = \mathrm{Det}\,\mathrm{RLine}(M, i, \mathrm{Line}(M, j))$.

(30) If $\mathrm{Det}\,M \neq 0_K$, then $M \cdot (\mathrm{Det}\,M^{-1} \cdot ($the matrix of cofactor $M)^{\mathrm{T}}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$.

(31) For all f, i such that $\mathrm{len}\,f = n$ and $i \in \mathrm{Seg}\,n$ holds (the matrix of cofactor $M)_{\square,i} \bullet f = \mathrm{LaplaceExpL}(\mathrm{RLine}(M^{\mathrm{T}}, i, f), i)$.

(32) If $i \in \mathrm{Seg}\,n$ and $j \in \mathrm{Seg}\,n$, then $\mathrm{Line}(($the matrix of cofactor $M)^{\mathrm{T}}, i) \cdot M_{\square,j} = \mathrm{Det}\,\mathrm{RLine}(M^{\mathrm{T}}, i, \mathrm{Line}(M^{\mathrm{T}}, j))$.

(33) If $\operatorname{Det} M \neq 0_K$, then $\operatorname{Det} M^{-1} \cdot (\text{the matrix of cofactor } M)^{\mathrm{T}} \cdot M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(34) M is invertible iff $\operatorname{Det} M \neq 0_K$.

(35) If $\operatorname{Det} M \neq 0_K$, then $M^{\smile} = \operatorname{Det} M^{-1} \cdot (\text{the matrix of cofactor } M)^{\mathrm{T}}$.

(36) Let M be a matrix over K of dimension n. Suppose M is invertible. Let given i, j. If $\langle i, j \rangle \in$ the indices of $M^{\smile}$, then $M^{\smile}{}_{i,j} = \operatorname{Det} M^{-1} \cdot \operatorname{power}_K(-\mathbf{1}_K, i+j) \cdot \operatorname{Minor}(M, j, i)$.

(37) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be matrices over K. Suppose $\operatorname{len} x = n$ and $A \cdot x = b$. Then $x = A^{\smile} \cdot b$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceCol}(A, i, b_{\square,j})$.

(38) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be matrices over K. Suppose $\operatorname{width} x = n$ and $x \cdot A = b$. Then $x = b \cdot A^{\smile}$ and for all i, j such that $\langle i, j \rangle \in$ the indices of x holds $x_{i,j} = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceLine}(A, j, \operatorname{Line}(b, i))$.

6. Product by a Vector

Let D be a non empty set and let f be a finite sequence of elements of D. Then $\langle f \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} f$.

Let us consider K, let M be a matrix over K, and let f be a finite sequence of elements of K. The functor $M \cdot f$ yielding a matrix over K is defined by:

(Def. 9) $M \cdot f = M \cdot \langle f \rangle^{\mathrm{T}}$.

The functor $f \cdot M$ yields a matrix over K and is defined by:

(Def. 10) $f \cdot M = \langle f \rangle \cdot M$.

Next we state two propositions:

(39) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be finite sequences of elements of K. Suppose $\operatorname{len} x = n$ and $A \cdot x = \langle b \rangle^{\mathrm{T}}$. Then $\langle x \rangle^{\mathrm{T}} = A^{\smile} \cdot b$ and for every i such that $i \in \operatorname{Seg} n$ holds $x(i) = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceCol}(A, i, b)$.

(40) Let A be a matrix over K of dimension n. Suppose $\operatorname{Det} A \neq 0_K$. Let x, b be finite sequences of elements of K. Suppose $\operatorname{len} x = n$ and $x \cdot A = \langle b \rangle$. Then $\langle x \rangle = b \cdot A^{\smile}$ and for every i such that $i \in \operatorname{Seg} n$ holds $x(i) = \operatorname{Det} A^{-1} \cdot \operatorname{Det} \operatorname{ReplaceLine}(A, i, b)$.

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