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Recognizing Chordal Graphs: Lex BFS and MCS[1]

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Summary. We are formalizing the algorithm for recognizing chordal graphs by lexicographic breadth-first search as presented in [13, Section 3 of Chapter 4, pp. 81–84]. Then we follow with a formalization of another algorithm serving the same end but based on maximum cardinality search as presented by Tarjan and Yannakakis [25].

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The notation and terminology used in this paper are introduced in the following articles: [28], [11], [26], [32], [33], [35], [30], [10], [7], [8], [20], [29], [4], [2], [14], [23], [12], [3], [6], [9], [18], [15], [19], [16], [17], [24], [21], [1], [5], [31], [27], [22], and [34].

1. Preliminaries

The following propositions are true:

(1) Let A, B be elements of $\mathbb{N}$, X be a non empty set, and F be a function from $\mathbb{N}$ into X. If F is one-to-one, then $\overline{\overline{\{F(w); w \text{ ranges over elements of } \mathbb{N}\colon A \leq w \ \wedge \ w \leq A + B\}}} = B + 1$.

(2) For all natural numbers n, m, k such that $m \leq k$ and $n < m$ holds $k -' m < k -' n$.

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(3) For all natural numbers n, k such that $n < k$ holds $(k -' (n+1)) + 1 = k -' n$.

(4) For all natural numbers n, m, k such that $k \neq 0$ holds $(n + m \cdot k) \div k = (n \div k) + m$.

Let S be a set. We say that S has finite elements if and only if:

(Def. 1) Every element of S is finite.

Let us note that there exists a set which is non empty and has finite elements and there exists a subset of $2^{\mathbb{N}}$ which is non empty and finite and has finite elements.

Let S be a set with finite elements. One can check that every element of S is finite.

Let f, g be functions. The functor $f[\cup]g$ yielding a function is defined by:

(Def. 2) $\mathrm{dom}(f[\cup]g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds $(f[\cup]g)(x) = f(x) \cup g(x)$.

The following three propositions are true:

(5) For all natural numbers m, n, k holds $m \in \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)$ iff $k -' n < m$ and $m \leq k$.

(6) For all natural numbers n, k, m such that $n \leq m$ holds $\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n) \subseteq \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' m)$.

(7) For all natural numbers n, k such that $n < k$ holds $(\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)) \cup \{k -' n\} = \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' (n+1))$.

Let f be a binary relation. We say that f is natsubset yielding if and only if:

(Def. 3) $\mathrm{rng}\, f \subseteq 2^{\mathbb{N}}$.

Let us mention that there exists a function which is finite-yielding and natsubset yielding.

Let f be a finite-yielding natsubset yielding function and let x be a set. Then $f(x)$ is a finite subset of $\mathbb{N}$.

One can prove the following proposition

(8) For every ordinal number X and for all finite subsets a, b of X such that $a \neq b$ holds $(a, 1)\text{-bag} \neq (b, 1)\text{-bag}$.

Let F be a natural-yielding function, let S be a set, and let k be a natural number. The functor $F.\mathrm{incSubset}(S, k)$ yielding a natural-yielding function is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{dom}(F.\mathrm{incSubset}(S, k)) = \mathrm{dom}\, F$, and

(ii) for every set y holds if $y \in S$ and $y \in \mathrm{dom}\, F$, then $(F.\mathrm{incSubset}(S, k))(y) = F(y) + k$ and if $y \notin S$, then $(F.\mathrm{incSubset}(S, k))(y) = F(y)$.

Let n be an ordinal number, let T be a connected term order of n, and let B be a non empty finite subset of Bags n. The functor $\max(B,T)$ yields a bag of n and is defined as follows:

(Def. 5) $\max(B,T) \in B$ and for every bag x of n such that $x \in B$ holds $x \leq_T \max(B,T)$.

Let O be an ordinal number. Observe that InvLexOrder O is connected.

2. Miscellany on Graphs

Let G be a graph. Note that there exists a vertex sequence of G which is non empty and one-to-one.

Let G be a graph and let V be a non empty vertex sequence of G. A walk of G is called a walk of V if:

(Def. 6) It.vertexSeq() $= V$.

Let G be a graph and let V be a non empty one-to-one vertex sequence of G. One can check that every walk of V is path-like.

We now state two propositions:

(9) For every graph G and for all walks W_1, W_2 of G such that W_1 is trivial and W_1.last() $= W_2$.first() holds W_1.append(W_2) $= W_2$.

(10) Let G, H be graphs, A, B, C be sets, G_1 be a subgraph of G induced by A, H_1 be a subgraph of H induced by B, G_2 be a subgraph of G_1 induced by C, and H_2 be a subgraph of H_1 induced by C. Suppose $G =_G H$ and $A \subseteq B$ and $C \subseteq A$ and C is a non empty subset of the vertices of G. Then $G_2 =_G H_2$.

Let G be a v-graph. We say that G is natural v-labeled if and only if:

(Def. 7) The vlabel of G is natural-yielding.

3. Graphs with Two Vertex Labels

The natural number V2-LabelSelector is defined by:

(Def. 8) V2-LabelSelector $= 8$.

Let G be a graph structure. We say that G is v2-labeled if and only if:

(Def. 9) V2-LabelSelector $\in \operatorname{dom} G$ and there exists a function f such that G(V2-LabelSelector) $= f$ and $\operatorname{dom} f \subseteq$ the vertices of G.

Let us note that there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph is a v2-labeled graph. A vv-graph is a vlabeled v2-labeled graph.

Let G be a v2-graph. The v2-label of G yields a function and is defined as follows:

(Def. 10) The v2-label of $G = G(\text{V2-LabelSelector})$.

Next we state the proposition

(11) For every v2-graph G holds dom (the v2-label of G) $\subseteq$ the vertices of G.

Let G be a graph and let X be a set. Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is graph-like.

We now state the proposition

(12) For every graph G and for every set X holds $G.\text{set}(\text{V2-LabelSelector}, X) =_G G$.

Let G be a finite graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is finite.

Let G be a loopless graph and let X be a set.

Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is loopless.

Let G be a trivial graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is trivial.

Let G be a non trivial graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non trivial.

Let G be a non-multi graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-multi.

Let G be a non-directed-multi graph and let X be a set. One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-directed-multi.

Let G be a connected graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is connected.

Let G be an acyclic graph and let X be a set.

One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is acyclic.

Let G be a v-graph and let X be a set.

One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is vlabeled.

Let G be a e-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is elabeled.

Let G be a w-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is weighted.

Let G be a v2-graph and let X be a set.

One can verify that $G.\text{set}(\text{VLabelSelector}, X)$ is v2-labeled.

Let G be a graph, let Y be a set, and let X be a partial function from the vertices of G to Y. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph and let X be a many sorted set indexed by the vertices of G. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph. One can verify that $G.\text{set}(\text{V2-LabelSelector}, \emptyset)$ is v2-labeled.

Let G be a v2-graph. We say that G is natural v2-labeled if and only if:

(Def. 11) The v2-label of G is natural-yielding.

We say that G is finite v2-labeled if and only if:

(Def. 12) The v2-label of G is finite-yielding.

We say that G is natsubset v2-labeled if and only if:

(Def. 13) The v2-label of G is natsubset yielding.

One can check that there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let G be a natural v-labeled v-graph. Observe that the vlabel of G is natural-yielding.

Let G be a natural v2-labeled v2-graph. Observe that the v2-label of G is natural-yielding.

Let G be a finite v2-labeled v2-graph. Observe that the v2-label of G is finite-yielding.

Let G be a natsubset v2-labeled v2-graph. One can verify that the v2-label of G is natsubset yielding.

Let G be a vv-graph and let v, x be sets. One can check that G.labelVertex(v, x) is v2-labeled.

Next we state the proposition

(13) For every vv-graph G and for all sets v, x holds the v2-label of $G =$ the v2-label of G.labelVertex(v, x).

Let G be a natural v-labeled vv-graph, let v be a set, and let x be a natural number. Observe that G.labelVertex(v, x) is natural v-labeled.

Let G be a natural v2-labeled vv-graph, let v be a set, and let x be a natural number. Observe that G.labelVertex(v, x) is natural v2-labeled.

Let G be a finite v2-labeled vv-graph, let v be a set, and let x be a natural number. Note that G.labelVertex(v, x) is finite v2-labeled.

Let G be a natsubset v2-labeled vv-graph, let v be a set, and let x be a natural number. One can check that G.labelVertex(v, x) is natsubset v2-labeled.

Let G be a graph. Note that there exists a subgraph of G which is vlabeled and v2-labeled.

Let G be a v2-graph and let G_2 be a v2-labeled subgraph of G. We say that G_2 inherits v2-label if and only if:

(Def. 14) The v2-label of $G_2 =$ (the v2-label of G)$\restriction$(the vertices of G_2).

Let G be a v2-graph. Note that there exists a v2-labeled subgraph of G which inherits v2-label.

Let G be a v2-graph. A v2-subgraph of G is a v2-labeled subgraph of G inheriting v2-label.

Let G be a vv-graph. Note that there exists a vlabeled v2-labeled subgraph of G which inherits vlabel and v2-label.

Let G be a vv-graph. A vv-subgraph of G is a vlabeled v2-labeled subgraph of G inheriting vlabel and v2-label.

Let G be a natural v-labeled v-graph. Note that every v-subgraph of G is natural v-labeled.

Let G be a graph and let V, E be sets. Observe that there exists a subgraph of G induced by V and E which is weighted, elabeled, vlabeled, and v2-labeled.

Let G be a vv-graph and let V, E be sets. Observe that there exists a vlabeled v2-labeled subgraph of G induced by V and E which inherits vlabel and v2-label.

Let G be a vv-graph and let V, E be sets. A (V, E)-induced vv-subgraph of G is a vlabeled v2-labeled subgraph of G induced by V and E inheriting vlabel and v2-label.

Let G be a vv-graph and let V be a set. A V-induced vv-subgraph of G is a $(V, G.\text{edgesBetween}(V))$-induced vv-subgraph of G.

4. More on Graph Sequences

Let s be a many sorted set indexed by $\mathbb{N}$. We say that s is iterative if and only if:

(Def. 15) For all natural numbers k, n such that $s(k) = s(n)$ holds $s(k+1) = s(n+1)$.

Let G_3 be a many sorted set indexed by $\mathbb{N}$. We say that G_3 is eventually constant if and only if:

(Def. 16) There exists a natural number n such that for every natural number m such that $n \leq m$ holds $G_3(n) = G_3(m)$.

Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is halting, iterative, and eventually constant.

The following proposition is true

(14) For every many sorted set G_4 indexed by $\mathbb{N}$ such that G_4 is halting and iterative holds G_4 is eventually constant.

One can check that every many sorted set indexed by $\mathbb{N}$ which is halting and iterative is also eventually constant.

The following proposition is true

(15) For every many sorted set G_4 indexed by $\mathbb{N}$ such that G_4 is eventually constant holds G_4 is halting.

Let us mention that every many sorted set indexed by $\mathbb{N}$ which is eventually constant is also halting.

One can prove the following two propositions:

(16) Let G_4 be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and n be a natural number. If $G_4.\text{Lifespan}() \leq n$, then $G_4(G_4.\text{Lifespan}()) = G_4(n)$.

(17) Let G_4 be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and n, m be natural numbers. If G_4.Lifespan() $\leq n$ and $n \leq m$, then $G_4(m) = G_4(n)$.

Let G_3 be a v-graph sequence. We say that G_3 is natural v-labeled if and only if:

(Def. 17) For every natural number x holds $G_3(x)$ is natural v-labeled.

Let G_3 be a graph sequence. We say that G_3 is chordal if and only if:

(Def. 18) For every natural number x holds $G_3(x)$ is chordal.

We say that G_3 has fixed vertices if and only if:

(Def. 19) For all natural numbers n, m holds the vertices of $G_3(n) =$ the vertices of $G_3(m)$.

We say that G_3 is v2-labeled if and only if:

(Def. 20) For every natural number x holds $G_3(x)$ is v2-labeled.

Let us observe that there exists a graph sequence which is weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph sequence is a v2-labeled graph sequence. A vv-graph sequence is a vlabeled v2-labeled graph sequence.

Let G_5 be a v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is v2-labeled.

Let G_5 be a v2-graph sequence. We say that G_5 is natural v2-labeled if and only if:

(Def. 21) For every natural number x holds $G_5(x)$ is natural v2-labeled.

We say that G_5 is finite v2-labeled if and only if:

(Def. 22) For every natural number x holds $G_5(x)$ is finite v2-labeled.

We say that G_5 is natsubset v2-labeled if and only if:

(Def. 23) For every natural number x holds $G_5(x)$ is natsubset v2-labeled.

Let us mention that there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let G_4 be a v-graph sequence and let x be a natural number. Then $G_4(x)$ is a v-graph.

Let G_5 be a natural v-labeled v-graph sequence and let x be a natural number. Observe that $G_5(x)$ is natural v-labeled.

Let G_5 be a natural v2-labeled v2-graph sequence and let x be a natural number. One can check that $G_5(x)$ is natural v2-labeled.

Let G_5 be a finite v2-labeled v2-graph sequence and let x be a natural number. One can verify that $G_5(x)$ is finite v2-labeled.

Let G_5 be a natsubset v2-labeled v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is natsubset v2-labeled.

Let G_5 be a chordal graph sequence and let x be a natural number. One can check that $G_5(x)$ is chordal.

Let G_4 be a v-graph sequence and let n be a natural number. Then $G_4(n)$ is a v-graph.

Let G_4 be a finite v-graph sequence and let n be a natural number. One can check that $G_4(n)$ is finite.

Let G_4 be a vv-graph sequence and let n be a natural number. Then $G_4(n)$ is a vv-graph.

Let G_4 be a finite vv-graph sequence and let n be a natural number. One can verify that $G_4(n)$ is finite.

Let G_4 be a chordal vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is chordal.

Let G_4 be a natural v-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natural v-labeled.

Let G_4 be a finite v2-labeled vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is finite v2-labeled.

Let G_4 be a natsubset v2-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natsubset v2-labeled.

Let G_4 be a natural v2-labeled vv-graph sequence and let n be a natural number. Observe that $G_4(n)$ is natural v2-labeled.

5. Vertices Numbering Sequences

Let G_3 be a v-graph sequence. We say that G_3 has initially empty v-label if and only if:

(Def. 24) The vlabel of $G_3(0) = \emptyset$.

We say that G_3 is adding one at a step if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let n be a natural number. Suppose $n < G_3$.Lifespan(). Then there exists a set w such that $w \notin \mathrm{dom}\,$(the vlabel of $G_3(n)$) and the vlabel of $G_3(n+1) =$ (the vlabel of $G_3(n)$)$+\cdot(w \dot{\longmapsto} (G_3.\mathrm{Lifespan}() -' n))$.

Let G_3 be a v-graph sequence. We say that G_3 is v-label numbering if and only if the condition (Def. 26) is satisfied.

(Def. 26) G_3 is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

One can check that there exists a v-graph sequence which is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

Let us observe that there exists a v-graph sequence which is v-label numbering.

One can check the following observations:

* every v-graph sequence which is v-label numbering is also iterative,
* every v-graph sequence which is v-label numbering is also eventually constant,
* every v-graph sequence which is v-label numbering is also finite,
* every v-graph sequence which is v-label numbering has also fixed vertices,
* every v-graph sequence which is v-label numbering is also natural v-labeled,
* every v-graph sequence which is v-label numbering has also initially empty v-label, and
* every v-graph sequence which is v-label numbering is also adding one at a step.

A v-label numbering sequence is a v-label numbering v-graph sequence.

Let G_3 be a v-label numbering sequence and let n be a natural number. The functor G_3 .PickedAt n yields a set and is defined by:

(Def. 27)(i) G_3 .PickedAt n = choose(the vertices of $G_3(0)$) if $n \geq G_3$.Lifespan(),

(ii) G_3 .PickedAt $n \notin$ dom (the vlabel of $G_3(n)$) and the vlabel of $G_3(n+1)$ = (the vlabel of $G_3(n)$)$+\cdot$((G_3 .PickedAt n)$\dot{\longmapsto}$(G_3.Lifespan() $-'$ n)), otherwise.

The following propositions are true:

(18) Let G_3 be a v-label numbering sequence and n be a natural number. If $n <$ G_3.Lifespan(), then G_3 .PickedAt $n \in G_3(n+1)$.labeledV() and $G_3(n+1)$.labeledV() = $G_3(n)$.labeledV() $\cup$ {G_3 .PickedAt n}.

(19) Let G_3 be a v-label numbering sequence and n be a natural number. If $n <$ G_3.Lifespan(), then (the vlabel of $G_3(n+1)$)(G_3 .PickedAt n) = G_3.Lifespan() $-'$ n.

(20) For every v-label numbering sequence G_3 and for every natural number n such that $n \leq G_3$.Lifespan() holds card($G_3(n)$.labeledV()) = n.

(21) For every v-label numbering sequence G_3 and for every natural number n holds rng (the vlabel of $G_3(n)$) = Seg(G_3.Lifespan()) $\setminus$ Seg(G_3.Lifespan() $-'$ n).

(22) Let G_3 be a v-label numbering sequence, n be a natural number, and x be a set. Then (the vlabel of $G_3(n)$)(x) $\leq G_3$.Lifespan() and if $x \in$ $G_3(n)$.labeledV(), then $1 \leq$ (the vlabel of $G_3(n)$)(x).

(23) Let G_3 be a v-label numbering sequence and n, m be natural numbers. Suppose G_3.Lifespan() $-'$ $n < m$ and $m \leq G_3$.Lifespan(). Then there exists a vertex v of $G_3(n)$ such that $v \in G_3(n)$.labeledV() and (the vlabel

of $G_3(n))(v) = m$.

(24) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $m \leq n$, then the vlabel of $G_3(m) \subseteq$ the vlabel of $G_3(n)$.

(25) For every v-label numbering sequence G_3 and for every natural number n holds the vlabel of $G_3(n)$ is one-to-one.

(26) Let G_3 be a v-label numbering sequence, m, n be natural numbers, and v be a set. Suppose $v \in G_3(m)$.labeledV() and $v \in G_3(n)$.labeledV(). Then (the vlabel of $G_3(m))(v) =$ (the vlabel of $G_3(n))(v)$.

(27) Let G_3 be a v-label numbering sequence, v be a set, and m, n be natural numbers. If $v \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(v) = n$, then G_3 .PickedAt(G_3.Lifespan() $-'$ n) $= v$.

(28) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $n < G_3$.Lifespan() and $n < m$, then G_3 .PickedAt $n \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(G_3$ .PickedAt $n) = G_3$.Lifespan() $-'$ n.

(29) Let G_3 be a v-label numbering sequence, m be a natural number, and v be a set. Suppose $v \in G_3(m)$.labeledV(). Then G_3.Lifespan() $-'$(the vlabel of $G_3(m))(v) < m$ and G_3.Lifespan() $-'$ $m <$ (the vlabel of $G_3(m))(v)$.

(30) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $b \in G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(a))$.labeledV().

(31) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $a \notin G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(b))$.labeledV().

6. Lexicographical Breadth-First Search

Let G be a graph. The functor LexBFS:Init G yields a natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 28) LexBFS:Init G = G.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of G) $\longmapsto \emptyset$).

Let G be a finite graph. Then LexBFS:Init G is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph. Let us assume that dom (the v2-label of G) = the vertices of G. The functor LexBFS:PickUnnumbered G yields a vertex of G and is defined by:

(Def. 29)(i) LexBFS:PickUnnumbered G = choose(the vertices of G) if dom (the vlabel of G) = the vertices of G,

(ii) there exists a non empty finite subset S of $2^{\mathbb{N}}$ and there exists a non empty finite subset B of Bags $\mathbb{N}$ and there exists a function F such that $S =$ rng F and $F =$ (the v2-label of G)$\restriction$((the vertices of G) $\setminus$ dom (the vlabel of G)) and for every finite subset x of $\mathbb{N}$ such that $x \in S$ holds $(x, 1)$-bag $\in B$ and for every set x such that $x \in B$ there exists a finite subset y of $\mathbb{N}$ such that $y \in S$ and $x = (y, 1)$-bag and LexBFS:PickUnnumbered $G =$ choose(F^{-1}({support max(B, InvLexOrder $\mathbb{N}$)})), otherwise.

Let G be a vv-graph, let v be a set, and let k be a natural number. The functor LexBFS:LabelAdjacent(G, v, k) yielding a vv-graph is defined as follows:

(Def. 30) LexBFS:LabelAdjacent$(G, v, k) = G$.set(V2-LabelSelector, (the v2-label of G)$[\cup]$((G.adjacentSet($\{v\}$)) $\setminus$ dom (the vlabel of G) $\longmapsto \{k\}$)).

Next we state four propositions:

(32) Let G be a vv-graph, v, x be sets, and k be a natural number. If $x \notin G$.adjacentSet($\{v\}$), then (the v2-label of G)(x) = (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x).

(33) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in$ dom (the vlabel of G). Then (the v2-label of G)(x) = (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x).

(34) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in G$.adjacentSet($\{v\}$) and $x \notin$ dom (the vlabel of G). Then (the v2-label of LexBFS:LabelAdjacent(G, v, k))(x) = (the v2-label of G)$(x) \cup \{k\}$.

(35) Let G be a vv-graph, v be a set, and k be a natural number. Suppose dom (the v2-label of G) = the vertices of G. Then dom (the v2-label of LexBFS:LabelAdjacent(G, v, k)) = the vertices of G.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G, and let k be a natural number. Then LexBFS:LabelAdjacent(G, v, k) is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G, and let n be a natural number. The functor LexBFS:Update(G, v, n) yielding a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph is defined by:

(Def. 31) LexBFS:Update$(G, v, n) =$
LexBFS:LabelAdjacent(G.labelVertex(v, G.order()$-'n$), v, G.order()$-'n$).

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph. The functor LexBFS:Step G yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 32) $$\text{LexBFS:Step}\,G = \begin{cases} G, & \text{if } G.\text{order}() \le \text{card dom (the vlabel of } G\text{)}, \\ \text{LexBFS:Update}(G, \text{LexBFS:PickUnnumbered}\,G, & \\ \quad \text{card dom (the vlabel of } G\text{))}, & \text{otherwise.} \end{cases}$$

Let G be a finite graph. The functor LexBFS:CSeq G yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph sequence and is defined by:

(Def. 33) (LexBFS:CSeq G)(0) = LexBFS:Init G and for every natural number n holds (LexBFS:CSeq G)($n+1$) = LexBFS:Step(LexBFS:CSeq G)(n).

We now state the proposition

(36) For every finite graph G holds LexBFS:CSeq G is iterative.

Let G be a finite graph. Observe that LexBFS:CSeq G is iterative.

Next we state a number of propositions:

(37) For every graph G holds the vlabel of LexBFS:Init $G = \emptyset$.

(38) Let G be a graph and v be a set. Then dom (the v2-label of LexBFS:Init G) = the vertices of G and (the v2-label of LexBFS:Init G)(v) = $\emptyset$.

(39) For every graph G holds $G =_G$ LexBFS:Init G.

(40) Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph and x be a set. Suppose that
  (i) $x \notin$ dom (the vlabel of G),
  (ii) dom (the v2-label of G) = the vertices of G, and
  (iii) dom (the vlabel of G) $\neq$ the vertices of G.

Then ((the v2-label of G)(x), 1) -bag $\leq_{\mathrm{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of G)(LexBFS:PickUnnumbered G), 1) -bag .

(41) Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph. Suppose dom (the v2-label of G) = the vertices of G and dom (the vlabel of G) $\neq$ the vertices of G. Then LexBFS:PickUnnumbered $G \notin$ dom (the vlabel of G).

(42) For every finite graph G and for every natural number n holds (LexBFS:CSeq G)(n) $=_G G$.

(43) For every finite graph G and for all natural numbers m, n holds (LexBFS:CSeq G)(m) $=_G$ (LexBFS:CSeq G)(n).

(44) Let G be a finite graph and n be a natural number. Suppose card dom (the vlabel of (LexBFS:CSeq G)(n)) < G.order(). Then the vlabel of (LexBFS:CSeq G)($n+1$) = (the vlabel of (LexBFS:CSeq G)(n))$+\cdot$(LexBFS:PickUnnumbered(LexBFS:CSeq G)(n) $\dot{\longmapsto}$(G.order() $-'$ card dom (the vlabel of (LexBFS:CSeq G)(n)))).

(45) For every finite graph G and for every natural number n holds dom (the v2-label of (LexBFS:CSeq G)(n)) = the vertices of (LexBFS:CSeq G)(n).

(46) For every finite graph G and for every natural number n such that $n \leq G$.order() holds card dom (the vlabel of (LexBFS:CSeq G)(n)) = n.

(47) For every finite graph G and for every natural number n such that G.order() $\leq n$ holds (LexBFS:CSeq G)(G.order()) =

(LexBFS:CSeq G)(n).

(48) For every finite graph G and for all natural numbers m, n such that G.order() $\leq m$ and $m \leq n$ holds (LexBFS:CSeq G)(m) = (LexBFS:CSeq G)(n).

(49) For every finite graph G holds LexBFS:CSeq G is eventually constant.

Let G be a finite graph. Note that LexBFS:CSeq G is eventually constant.

We now state two propositions:

(50) Let G be a finite graph and n be a natural number. Then dom (the vlabel of (LexBFS:CSeq G)(n)) = the vertices of (LexBFS:CSeq G)(n) if and only if G.order() $\leq n$.

(51) For every finite graph G holds (LexBFS:CSeq G).Lifespan() = G.order().

Let G be a finite chordal graph and let i be a natural number. One can check that (LexBFS:CSeq G)(i) is chordal.

Let G be a finite chordal graph. One can check that LexBFS:CSeq G is chordal.

One can prove the following proposition

(52) For every finite graph G holds LexBFS:CSeq G is v-label numbering.

Let G be a finite graph. Note that LexBFS:CSeq G is v-label numbering.

We now state several propositions:

(53) For every finite graph G and for every natural number n such that $n <$ G.order() holds LexBFS:CSeq G .PickedAt n = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n).

(54) Let G be a finite graph and n be a natural number. Suppose $n <$ G.order(). Then there exists a vertex w of (LexBFS:CSeq G)(n) such that
  (i) w = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n), and
  (ii) for every set v holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (LexBFS:CSeq G)(n)), then (the v2-label of (LexBFS:CSeq G)($n+1$))(v) = (the v2-label of (LexBFS:CSeq G)(n))(v)$\cup\{G$.order()$-'n\}$ and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (LexBFS:CSeq G)(n)), then (the v2-label of (LexBFS:CSeq G)($n+1$))(v) = (the v2-label of (LexBFS:CSeq G)(n))(v).

(55) Let G be a finite graph, i be a natural number, and v be a set. Then (the v2-label of (LexBFS:CSeq G)(i))(v) $\subseteq$ Seg(G.order()) $\setminus$ Seg(G.order()$-'i$).

(56) Let G be a finite graph, x be a set, and i, j be natural numbers. If $i \leq j$, then (the v2-label of (LexBFS:CSeq G)(i))(x) $\subseteq$ (the v2-label of (LexBFS:CSeq G)(j))(x).

(57) Let G be a finite graph, m, n be natural numbers, and x, y be sets. Suppose $n <$ G.order() and $n < m$ and y = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n) and $x \notin$ dom (the vlabel of

(LexBFS:CSeq G)(n)) and $x \in G$.adjacentSet($\{y\}$). Then G.order() $-'\, n \in$ (the v2-label of (LexBFS:CSeq G)(m))(x).

(58) Let G be a finite graph and m, n be natural numbers. Suppose $m < n$. Let x be a set. Suppose G.order() $-'\, m \notin$ (the v2-label of (LexBFS:CSeq G)($m+1$))(x). Then G.order() $-'\, m \notin$ (the v2-label of (LexBFS:CSeq G)(n))(x).

(59) Let G be a finite graph and m, n, k be natural numbers. Suppose $k < n$ and $n \leq m$. Let x be a set. Suppose G.order() $-'\, k \notin$ (the v2-label of (LexBFS:CSeq G)(n))(x). Then G.order() $-'\, k \notin$ (the v2-label of (LexBFS:CSeq G)(m))(x).

(60) Let G be a finite graph, m, n be natural numbers, and x be a vertex of (LexBFS:CSeq G)(m). Suppose $n \in$ (the v2-label of (LexBFS:CSeq G)(m))(x). Then there exists a vertex y of (LexBFS:CSeq G)(m) such that LexBFS:PickUnnumbered(LexBFS:CSeq G) (G.order() $-'\, n$) $= y$ and $y \notin$ dom (the vlabel of (LexBFS:CSeq G)(G.order() $-'\, n$)) and $x \in G$.adjacentSet($\{y\}$).

Let G_4 be a finite natural v-labeled vv-graph sequence. Then G_4.Result() is a finite natural v-labeled vv-graph.

The following four propositions are true:

(61) For every finite graph G holds (LexBFS:CSeq G).Result().labeledV() = the vertices of G.

(62) For every finite graph G holds (the vlabel of (LexBFS:CSeq G).Result())$^{-1}$ is a vertex scheme of G.

(63) Let G be a finite graph, i, j be natural numbers, and a, b be vertices of (LexBFS:CSeq G)(i). Suppose that

(i) $a \in$ dom (the vlabel of (LexBFS:CSeq G)(i)),

(ii) $b \in$ dom (the vlabel of (LexBFS:CSeq G)(i)),

(iii) (the vlabel of (LexBFS:CSeq G)(i))(a) $<$ (the vlabel of (LexBFS:CSeq G)(i))(b), and

(iv) $j = G$.order() $-'$ (the vlabel of (LexBFS:CSeq G)(i))(b).

Then ((the v2-label of (LexBFS:CSeq G)(j))(a), 1)-bag $\leq_{\mathrm{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of (LexBFS:CSeq G)(j))(b), 1)-bag.

(64) Let G be a finite graph, i, j be natural numbers, and v be a vertex of (LexBFS:CSeq G)(i). Suppose $j \in$ (the v2-label of (LexBFS:CSeq G)(i))(v). Then there exists a vertex w of (LexBFS:CSeq G)(i) such that $w \in$ dom (the vlabel of (LexBFS:CSeq G)(i)) and (the vlabel of (LexBFS:CSeq G)(i))(w) $= j$ and $v \in G$.adjacentSet($\{w\}$).

Let G be a natural v-labeled v-graph. We say that G has property *L3* if and only if the condition (Def. 34) is satisfied.

(Def. 34) Let a, b, c be vertices of G. Suppose that $a \in \mathrm{dom}$ (the vlabel of G) and $b \in \mathrm{dom}$ (the vlabel of G) and $c \in \mathrm{dom}$ (the vlabel of G) and (the vlabel of G)(a) $<$ (the vlabel of G)(b) and (the vlabel of G)(b) $<$ (the vlabel of G)(c) and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that

(i) $d \in \mathrm{dom}$ (the vlabel of G),

(ii) (the vlabel of G)(c) $<$ (the vlabel of G)(d),

(iii) b and d are adjacent,

(iv) a and d are not adjacent, and

(v) for every vertex e of G such that $e \neq d$ and e and b are adjacent and e and a are not adjacent holds (the vlabel of G)(e) $<$ (the vlabel of G)(d).

One can prove the following three propositions:

(65) For every finite graph G and for every natural number n holds (LexBFS:CSeq G)(n) has property *L3*.

(66) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property *L3* and dom (the vlabel of G) = the vertices of G. Let V be a vertex scheme of G. If V^{-1} = the vlabel of G, then V is perfect.

(67) For every finite chordal vv-graph G holds (the vlabel of (LexBFS:CSeq G).Result())$^{-1}$ is a perfect vertex scheme of G.

7. The Maximum Cardinality Search Algorithm

Let G be a finite graph. The functor MCS:Init G yields a finite natural v-labeled natural v2-labeled vv-graph and is defined by:

(Def. 35) MCS:Init G = G.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of G) $\longmapsto 0$).

Let G be a finite natural v2-labeled vv-graph. Let us assume that dom (the v2-label of G) = the vertices of G. The functor MCS:PickUnnumbered G yields a vertex of G and is defined by:

(Def. 36)(i) MCS:PickUnnumbered G = choose(the vertices of G) if dom (the vlabel of G) = the vertices of G,

(ii) there exists a finite non empty natural-membered set S and there exists a function F such that $S = \mathrm{rng}\, F$ and F = (the v2-label of G)$\restriction$((the vertices of G) $\setminus$ dom (the vlabel of G)) and MCS:PickUnnumbered G = choose($F^{-1}(\{\max S\})$), otherwise.

Let G be a finite natural v2-labeled vv-graph and let v be a set. The functor MCS:LabelAdjacent(G, v) yields a finite natural v2-labeled vv-graph and is defined by:

(Def. 37) MCS:LabelAdjacent(G, v) = G.set(V2-LabelSelector, (the v2-label of G) .incSubset((G.adjacentSet($\{v\}$)) $\setminus$ dom (the vlabel of G), 1)).

Let G be a finite natural v-labeled natural v2-labeled vv-graph and let v be a vertex of G. Then MCS:LabelAdjacent(G, v) is a finite natural v-labeled natural v2-labeled vv-graph.

Let G be a finite natural v-labeled natural v2-labeled vv-graph, let v be a vertex of G, and let n be a natural number. The functor MCS:Update(G, v, n) yielding a finite natural v-labeled natural v2-labeled vv-graph is defined as follows:

(Def. 38) MCS:Update(G, v, n) = MCS:LabelAdjacent(G.labelVertex(v, G.order()$-'$ $n), v$).

Let G be a finite natural v-labeled natural v2-labeled vv-graph. The functor MCS:Step G yielding a finite natural v-labeled natural v2-labeled vv-graph is defined by:

(Def. 39) $$\text{MCS:Step}\, G = \begin{cases} G, \text{ if } G.\text{order}() \leq \text{card dom (the vlabel of } G), \\ \text{MCS:Update}(G, \text{MCS:PickUnnumbered}\, G, \text{card dom} \\ \quad \text{(the vlabel of } G)), \text{ otherwise.} \end{cases}$$

Let G be a finite graph. The functor MCS:CSeq G yields a finite natural v-labeled natural v2-labeled vv-graph sequence and is defined by:

(Def. 40) (MCS:CSeq G)(0) = MCS:Init G and for every natural number n holds (MCS:CSeq G)$(n + 1)$ = MCS:Step(MCS:CSeq G)(n).

The following proposition is true

(68) For every finite graph G holds MCS:CSeq G is iterative.

Let G be a finite graph. Observe that MCS:CSeq G is iterative.

We now state a number of propositions:

(69) For every finite graph G holds the vlabel of MCS:Init $G = \emptyset$.

(70) Let G be a finite graph and v be a set. Then dom (the v2-label of MCS:Init G) = the vertices of G and (the v2-label of MCS:Init G)$(v) = 0$.

(71) For every finite graph G holds $G =_G$ MCS:Init G.

(72) Let G be a finite natural v2-labeled vv-graph and x be a set. Suppose that

(i) $x \notin$ dom (the vlabel of G),

(ii) dom (the v2-label of G) = the vertices of G, and

(iii) dom (the vlabel of G) $\neq$ the vertices of G.

Then (the v2-label of G)$(x) \leq$ (the v2-label of G)(MCS:PickUnnumbered G).

(73) Let G be a finite natural v2-labeled vv-graph. Suppose dom (the v2-label of G) = the vertices of G and dom (the vlabel of G) $\neq$ the vertices of G. Then MCS:PickUnnumbered $G \notin$ dom (the vlabel of G).

(74) Let G be a finite natural v2-labeled vv-graph and v, x be sets. If $x \notin G$.adjacentSet($\{v\}$), then (the v2-label of G)(x) = (the v2-label of

MCS:LabelAdjacent$(G, v))(x)$.

(75) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in$ dom (the vlabel of G). Then (the v2-label of $G)(x) =$ (the v2-label of MCS:LabelAdjacent$(G, v))(x)$.

(76) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in$ dom (the v2-label of G) and $x \in G$.adjacentSet($\{v\}$) and $x \notin$ dom (the vlabel of G). Then (the v2-label of MCS:LabelAdjacent$(G, v))(x) =$ (the v2-label of $G)(x) + 1$.

(77) Let G be a finite natural v2-labeled vv-graph and v be a set. Suppose dom (the v2-label of G) = the vertices of G. Then dom (the v2-label of MCS:LabelAdjacent(G, v)) = the vertices of G.

(78) For every finite graph G and for every natural number n holds (MCS:CSeq $G)(n) =_G G$.

(79) For every finite graph G and for all natural numbers m, n holds (MCS:CSeq $G)(m) =_G$ (MCS:CSeq $G)(n)$.

Let G be a finite chordal graph and let n be a natural number. Observe that (MCS:CSeq $G)(n)$ is chordal.

Let G be a finite chordal graph. Observe that MCS:CSeq G is chordal.

One can prove the following propositions:

(80) For every finite graph G and for every natural number n holds dom (the v2-label of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$.

(81) Let G be a finite graph and n be a natural number. Suppose card dom (the vlabel of (MCS:CSeq $G)(n)) < G$.order(). Then the vlabel of (MCS:CSeq $G)(n+1) =$ (the vlabel of (MCS:CSeq $G)(n)$) $+\cdot$(MCS:PickUnnumbered(MCS:CSeq $G)(n) \dot{\longmapsto} (G$.order()$-'$card dom (the vlabel of (MCS:CSeq $G)(n)$))).

(82) For every finite graph G and for every natural number n such that $n \leq G$.order() holds card dom (the vlabel of (MCS:CSeq $G)(n)) = n$.

(83) For every finite graph G and for every natural number n such that G.order() $\leq n$ holds (MCS:CSeq $G)(G$.order()) = (MCS:CSeq $G)(n)$.

(84) For every finite graph G and for all natural numbers m, n such that G.order() $\leq m$ and $m \leq n$ holds (MCS:CSeq $G)(m) =$ (MCS:CSeq $G)(n)$.

(85) For every finite graph G holds MCS:CSeq G is eventually constant.

Let G be a finite graph. Observe that MCS:CSeq G is eventually constant.

The following propositions are true:

(86) Let G be a finite graph and n be a natural number. Then dom (the vlabel of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$ if and only if G.order() $\leq n$.

(87) For every finite graph G holds (MCS:CSeq G).Lifespan() = G.order().

(88) For every finite graph G holds MCS:CSeq G is v-label numbering.

Let G be a finite graph. Note that MCS:CSeq G is v-label numbering.

Next we state three propositions:

(89) For every finite graph G and for every natural number n such that $n < G$.order() holds MCS:CSeq G .PickedAt $n =$ MCS:PickUnnumbered(MCS:CSeq G)(n).

(90) Let G be a finite graph and n be a natural number. Suppose $n < G$.order(). Then there exists a vertex w of (MCS:CSeq G)(n) such that

(i) $w =$ MCS:PickUnnumbered(MCS:CSeq G)(n), and

(ii) for every set v holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (MCS:CSeq G)(n)), then (the v2-label of (MCS:CSeq G)($n+1$))(v) = (the v2-label of (MCS:CSeq G)(n))(v) + 1 and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (MCS:CSeq G)(n)), then (the v2-label of (MCS:CSeq G)($n+1$))(v) = (the v2-label of (MCS:CSeq G)(n))(v).

(91) Let G be a finite graph, n be a natural number, and x be a set. Suppose $x \notin$ dom (the vlabel of (MCS:CSeq G)(n)). Then (the v2-label of (MCS:CSeq G)(n))(x) = card((G.adjacentSet($\{x\}$)) $\cap$ dom (the vlabel of (MCS:CSeq G)(n))).

Let G be a natural v-labeled v-graph. We say that G has property T if and only if the condition (Def. 41) is satisfied.

(Def. 41) Let a, b, c be vertices of G. Suppose that $a \in$ dom (the vlabel of G) and $b \in$ dom (the vlabel of G) and $c \in$ dom (the vlabel of G) and (the vlabel of G)(a) < (the vlabel of G)(b) and (the vlabel of G)(b) < (the vlabel of G)(c) and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that

(i) $d \in$ dom (the vlabel of G),

(ii) (the vlabel of G)(b) < (the vlabel of G)(d),

(iii) b and d are adjacent, and

(iv) a and d are not adjacent.

We now state three propositions:

(92) For every finite graph G and for every natural number n holds (MCS:CSeq G)(n) has property T.

(93) For every finite graph G holds (LexBFS:CSeq G).Result() has property T.

(94) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property T and dom (the vlabel of G) = the vertices of G. Let V be a vertex scheme of G. If V^{-1} = the vlabel of G, then V is perfect.

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