FORMALIZED MATHEMATICS
2006, Vol. **14**, No. **4**, Pages 161–169
DOI: 10.2478/v10037-006-0020-1

The Quaternion Numbers

Xiquan Liang
Qingdao University of Science and Technology
China

Fuguo Ge
Qingdao University of Science and Technology
China

Summary. In this article, we define the set $\mathbb{H}$ of quaternion numbers as the set of all ordered sequences $q = \langle x, y, w, z \rangle$ where x,y,w and z are real numbers. The addition, difference and multiplication of the quaternion numbers are also defined. We define the real and imaginary parts of q and denote this by $x = \Re(q)$, $y = \Im_1(q)$, $w = \Im_2(q)$, $z = \Im_3(q)$. We define the addition, difference, multiplication again and denote this operation by real and three imaginary parts. We define the conjugate of q denoted by $q*'$ and the absolute value of q denoted by $|q|$. We also give some properties of quaternion numbers.

MML identifier: `QUATERNI`, version: `7.8.03 4.75.958`

The articles [14], [16], [2], [1], [12], [17], [4], [5], [6], [13], [3], [11], [7], [8], [15], [18], [9], and [10] provide the terminology and notation for this paper.

We use the following convention: a, b, c, d, x, y, w, z, x_1, x_2, x_3, x_4 denote sets and A denotes a non empty set.

The functor $\mathbb{H}$ is defined by:

(Def. 1) $\mathbb{H} = (\mathbb{R}^4 \setminus \{x; x$ ranges over elements of $\mathbb{R}^4$: $x(2) = 0 \wedge x(3) = 0\}) \cup \mathbb{C}$.

Let x be a number. We say that x is quaternion if and only if:

(Def. 2) $x \in \mathbb{H}$.

Let us observe that $\mathbb{H}$ is non empty.

Let us consider x, y, w, z, a, b, c, d. The functor $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ yields a set and is defined as follows:

(Def. 3) $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = [x \longmapsto a, y \longmapsto b] {+\cdot} [w \longmapsto c, z \longmapsto d]$.

Let us consider x, y, w, z, a, b, c, d. Note that $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is function-like and relation-like.

Next we state several propositions:

ISSN 1426–2630

(1) $\mathrm{dom}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{x, y, w, z\}$.

(2) $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] \subseteq \{a, b, c, d\}$.

(3) Suppose x, y, w, z are mutually different. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](x) = a$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](y) = b$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](w) = c$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](z) = d$.

(4) If x, y, w, z are mutually different, then $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{a, b, c, d\}$.

(5) $\{x_1, x_2, x_3, x_4\} \subseteq X$ iff $x_1 \in X$ and $x_2 \in X$ and $x_3 \in X$ and $x_4 \in X$.

Let us consider A, x, y, w, z and let a, b, c, d be elements of A. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is a function from $\{x, y, w, z\}$ into A.

The functor j is defined by:

(Def. 4) $j = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 0]$.

The functor k is defined by:

(Def. 5) $k = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 0, 3 \mapsto 1]$.

One can check the following observations:

* i is quaternion,
* j is quaternion, and
* k is quaternion.

Let us observe that there exists a number which is quaternion.

Let us mention that every element of $\mathbb{H}$ is quaternion.

Let x, y, w, z be elements of $\mathbb{R}$. The functor $\langle x, y, w, z\rangle_{\mathbb{H}}$ yields an element of $\mathbb{H}$ and is defined as follows:

(Def. 6) $\langle x, y, w, z\rangle_{\mathbb{H}} = \begin{cases} x + yi, & \text{if } w = 0 \text{ and } z = 0, \\ [0 \mapsto x, 1 \mapsto y, 2 \mapsto w, 3 \mapsto z], & \text{otherwise.} \end{cases}$

Next we state three propositions:

(6) Let a, b, c, d, e, i, j, k be sets and g be a function. Suppose $a \neq b$ and $c \neq d$ and $\mathrm{dom}\, g = \{a, b, c, d\}$ and $g(a) = e$ and $g(b) = i$ and $g(c) = j$ and $g(d) = k$. Then $g = [a \mapsto e, b \mapsto i, c \mapsto j, d \mapsto k]$.

(7) For every element g of $\mathbb{H}$ there exist elements r, s, t, u of $\mathbb{R}$ such that $g = \langle r, s, t, u\rangle_{\mathbb{H}}$.

(8) If a, c, x, w are mutually different, then $[a \mapsto b, c \mapsto d, x \mapsto y, w \mapsto z] = \{\langle a, b\rangle, \langle c, d\rangle, \langle x, y\rangle, \langle w, z\rangle\}$.

We adopt the following convention: a, b, c, d are elements of $\mathbb{R}$ and r, s, t are elements of $\mathbb{Q}_+$.

One can prove the following four propositions:

(9) Let A be a subset of $\mathbb{Q}_+$. Suppose there exists t such that $t \in A$ and $t \neq \emptyset$ and for all r, s such that $r \in A$ and $s \leq r$ holds $s \in A$. Then there exist elements r_1, r_2, r_3, r_4, r_5 of $\mathbb{Q}_+$ such that

$r_1 \in A$ and $r_2 \in A$ and $r_3 \in A$ and $r_4 \in A$ and $r_5 \in A$ and $r_1 \neq r_2$ and $r_1 \neq r_3$ and $r_1 \neq r_4$ and $r_1 \neq r_5$ and $r_2 \neq r_3$ and $r_2 \neq r_4$ and $r_2 \neq r_5$ and $r_3 \neq r_4$ and $r_3 \neq r_5$ and $r_4 \neq r_5$.

(10) $[0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d] \notin \mathbb{C}$.

(11) Let a, b, c, d, x, y, z, w, x', y', z', w' be sets. Suppose a, b, c, d are mutually different and $[a \mapsto x, b \mapsto y, c \mapsto z, d \mapsto w] = [a \mapsto x', b \mapsto y', c \mapsto z', d \mapsto w']$. Then $x = x'$ and $y = y'$ and $z = z'$ and $w = w'$.

(12) For all elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $\langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}} = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ holds $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$.

Let x, y be quaternion numbers. The functor $x + y$ is defined by:

(Def. 7) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 \rangle_{\mathbb{H}}$.

Let us observe that the functor $x + y$ is commutative.

Let z be a quaternion number. The functor $-z$ yields a quaternion number and is defined by:

(Def. 8) $z + -z = 0$.

Let us observe that the functor $-z$ is involutive.

Let x, y be quaternion numbers. The functor $x - y$ is defined as follows:

(Def. 9) $x - y = x + -y$.

Let x, y be quaternion numbers. The functor $x \cdot y$ is defined by the condition (Def. 10).

(Def. 10) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 - x_4 \cdot y_4, (x_1 \cdot y_2 + x_2 \cdot y_1 + x_3 \cdot y_4) - x_4 \cdot y_3, (x_1 \cdot y_3 + y_1 \cdot x_3 + y_2 \cdot x_4) - y_4 \cdot x_2, (x_1 \cdot y_4 + x_4 \cdot y_1 + x_2 \cdot y_3) - x_3 \cdot y_2 \rangle_{\mathbb{H}}$.

Let z, z' be quaternion numbers. One can verify the following observations:

* $z + z'$ is quaternion,
* $z \cdot z'$ is quaternion, and
* $z - z'$ is quaternion.

j Is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 11) $j = \langle 0, 0, 1, 0 \rangle_{\mathbb{H}}$.

Then k is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 12) $k = \langle 0, 0, 0, 1 \rangle_{\mathbb{H}}$.

One can prove the following propositions:

(13) $i \cdot i = -1$.

(14) $j \cdot j = -1$.

(15) $k \cdot k = -1$.

(16) $i \cdot j = k$.

(17) $j \cdot k = i$.

(18) $k \cdot i = j$.

(19) $i \cdot j = -j \cdot i$.

(20) $j \cdot k = -k \cdot j$.

(21) $k \cdot i = -i \cdot k$.

Let z be a quaternion number. The functor $\Re(z)$ is defined as follows:

(Def. 13)(i) There exists a complex number z' such that $z = z'$ and $\Re(z) = \Re(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Re(z) = f(0)$, otherwise.

The functor $\Im_1(z)$ is defined by:

(Def. 14)(i) There exists a complex number z' such that $z = z'$ and $\Im_1(z) = \Im(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_1(z) = f(1)$, otherwise.

The functor $\Im_2(z)$ is defined as follows:

(Def. 15)(i) $\Im_2(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_2(z) = f(2)$, otherwise.

The functor $\Im_3(z)$ is defined by:

(Def. 16)(i) $\Im_3(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_3(z) = f(3)$, otherwise.

Let z be a quaternion number. One can check the following observations:

* $\Re(z)$ is real,
* $\Im_1(z)$ is real,
* $\Im_2(z)$ is real, and
* $\Im_3(z)$ is real.

Let z be a quaternion number. Then $\Re(z)$ is a real number. Then $\Im_1(z)$ is a real number. Then $\Im_2(z)$ is a real number. Then $\Im_3(z)$ is a real number.

One can prove the following two propositions:

(22) For every function f from 4 into $\mathbb{R}$ there exist a, b, c, d such that $f = [0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d]$.

(23) $\Re(\langle a, b, c, d \rangle_{\mathbb{H}}) = a$ and $\Im_1(\langle a, b, c, d \rangle_{\mathbb{H}}) = b$ and $\Im_2(\langle a, b, c, d \rangle_{\mathbb{H}}) = c$ and $\Im_3(\langle a, b, c, d \rangle_{\mathbb{H}}) = d$.

In the sequel z, z_1, z_2, z_3, z_4 denote quaternion numbers.

Next we state two propositions:

(24) $z = \langle \Re(z), \Im_1(z), \Im_2(z), \Im_3(z) \rangle_{\mathbb{H}}$.

(25) If $\Re(z_1) = \Re(z_2)$ and $\Im_1(z_1) = \Im_1(z_2)$ and $\Im_2(z_1) = \Im_2(z_2)$ and $\Im_3(z_1) = \Im_3(z_2)$, then $z_1 = z_2$.

The quaternion number $0_{\mathbb{H}}$ is defined as follows:

(Def. 17) $0_{\mathbb{H}} = 0$.

The quaternion number $1_{\mathbb{H}}$ is defined as follows:

(Def. 18) $1_{\mathbb{H}} = 1$.

One can prove the following propositions:

(26) If $\Re(z) = 0$ and $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $z = 0_{\mathbb{H}}$.

(27) If $z = 0$, then $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$.

(28) If $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$, then $z = 0_{\mathbb{H}}$.

(29) $\Re(1_{\mathbb{H}}) = 1$ and $\Im_1(1_{\mathbb{H}}) = 0$ and $\Im_2(1_{\mathbb{H}}) = 0$ and $\Im_3(1_{\mathbb{H}}) = 0$.

(30) $\Re(i) = 0$ and $\Im_1(i) = 1$ and $\Im_2(i) = 0$ and $\Im_3(i) = 0$.

(31) $\Re(j) = 0$ and $\Im_1(j) = 0$ and $\Im_2(j) = 1$ and $\Im_3(j) = 0$ and $\Re(k) = 0$ and $\Im_1(k) = 0$ and $\Im_2(k) = 0$ and $\Im_3(k) = 1$.

(32) $\Re(z_1 + z_2 + z_3 + z_4) = \Re(z_1) + \Re(z_2) + \Re(z_3) + \Re(z_4)$ and $\Im_1(z_1 + z_2 + z_3 + z_4) = \Im_1(z_1) + \Im_1(z_2) + \Im_1(z_3) + \Im_1(z_4)$ and $\Im_2(z_1 + z_2 + z_3 + z_4) = \Im_2(z_1) + \Im_2(z_2) + \Im_2(z_3) + \Im_2(z_4)$ and $\Im_3(z_1 + z_2 + z_3 + z_4) = \Im_3(z_1) + \Im_3(z_2) + \Im_3(z_3) + \Im_3(z_4)$.

In the sequel x denotes a real number.

We now state three propositions:

(33) If $z_1 = x$, then $\Re(z_1 \cdot i) = 0$ and $\Im_1(z_1 \cdot i) = x$ and $\Im_2(z_1 \cdot i) = 0$ and $\Im_3(z_1 \cdot i) = 0$.

(34) If $z_1 = x$, then $\Re(z_1 \cdot j) = 0$ and $\Im_1(z_1 \cdot j) = 0$ and $\Im_2(z_1 \cdot j) = x$ and $\Im_3(z_1 \cdot j) = 0$.

(35) If $z_1 = x$, then $\Re(z_1 \cdot k) = 0$ and $\Im_1(z_1 \cdot k) = 0$ and $\Im_2(z_1 \cdot k) = 0$ and $\Im_3(z_1 \cdot k) = x$.

Let x be a real number and let y be a quaternion number. The functor $x + y$ is defined as follows:

(Def. 19) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x + y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$.

Let x be a real number and let y be a quaternion number. The functor $x - y$ is defined by:

(Def. 20) $x - y = x + -y$.

Let x be a real number and let y be a quaternion number. The functor $x \cdot y$ is defined as follows:

(Def. 21) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4\rangle_{\mathbb{H}}$ and $x \cdot y = \langle x \cdot y_1, x \cdot y_2, x \cdot y_3, x \cdot y_4\rangle_{\mathbb{H}}$.

Let x be a real number and let z' be a quaternion number. One can verify the following observations:

* $x + z'$ is quaternion,
* $x \cdot z'$ is quaternion, and
* $x - z'$ is quaternion.

Let z_1, z_2 be quaternion numbers. Then $z_1 + z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 22) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + (\Im_1(z_1) + \Im_1(z_2)) \cdot i + (\Im_2(z_1) + \Im_2(z_2)) \cdot j + (\Im_3(z_1) + \Im_3(z_2)) \cdot k$.

The following proposition is true

(36) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im_1(z_1 + z_2) = \Im_1(z_1) + \Im_1(z_2)$ and $\Im_2(z_1 + z_2) = \Im_2(z_1) + \Im_2(z_2)$ and $\Im_3(z_1 + z_2) = \Im_3(z_1) + \Im_3(z_2)$.

Let z_1, z_2 be elements of $\mathbb{H}$. Then $z_1 \cdot z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 23) $z_1 \cdot z_2 = (\Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)) + ((\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)) \cdot i + ((\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)) \cdot j + ((\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)) \cdot k$.

We now state four propositions:

(37) $z = \Re(z) + \Im_1(z) \cdot i + \Im_2(z) \cdot j + \Im_3(z) \cdot k$.

(38) Suppose $\Im_1(z_1) = 0$ and $\Im_1(z_2) = 0$ and $\Im_2(z_1) = 0$ and $\Im_2(z_2) = 0$ and $\Im_3(z_1) = 0$ and $\Im_3(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(39) Suppose $\Re(z_1) = 0$ and $\Re(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = -\Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(40) $\Re(z \cdot z) = (\Re(z))^{\mathbf{2}} - (\Im_1(z))^{\mathbf{2}} - (\Im_2(z))^{\mathbf{2}} - (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_1(z))$ and $\Im_2(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_2(z))$ and $\Im_3(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_3(z))$.

Let z be a quaternion number. Then $-z$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 24) $-z = -\Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

The following proposition is true

(41) $\Re(-z) = -\Re(z)$ and $\Im_1(-z) = -\Im_1(z)$ and $\Im_2(-z) = -\Im_2(z)$ and $\Im_3(-z) = -\Im_3(z)$.

Let z_1, z_2 be quaternion numbers. Then $z_1 - z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 25) $z_1 - z_2 = (\Re(z_1) - \Re(z_2)) + (\Im_1(z_1) - \Im_1(z_2)) \cdot i + (\Im_2(z_1) - \Im_2(z_2)) \cdot j + (\Im_3(z_1) - \Im_3(z_2)) \cdot k$.

One can prove the following proposition

(42) $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im_1(z_1 - z_2) = \Im_1(z_1) - \Im_1(z_2)$ and $\Im_2(z_1 - z_2) = \Im_2(z_1) - \Im_2(z_2)$ and $\Im_3(z_1 - z_2) = \Im_3(z_1) - \Im_3(z_2)$.

Let z be a quaternion number. The functor $\overline{z}$ yielding a quaternion number is defined by:

(Def. 26) $\overline{z} = \Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

Let z be a quaternion number. Then $\overline{z}$ is an element of $\mathbb{H}$.

We now state a number of propositions:

(43) $\overline{z} = \langle \Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(44) $\Re(\overline{z}) = \Re(z)$ and $\Im_1(\overline{z}) = -\Im_1(z)$ and $\Im_2(\overline{z}) = -\Im_2(z)$ and $\Im_3(\overline{z}) = -\Im_3(z)$.

(45) If $z = 0$, then $\overline{z} = 0$.

(46) If $\overline{z} = 0$, then $z = 0$.

(47) $\overline{1_{\mathbb{H}}} = 1_{\mathbb{H}}$.

(48) $\Re(\overline{i}) = 0$ and $\Im_1(\overline{i}) = -1$ and $\Im_2(\overline{i}) = 0$ and $\Im_3(\overline{i}) = 0$.

(49) $\Re(\overline{j}) = 0$ and $\Im_1(\overline{j}) = 0$ and $\Im_2(\overline{j}) = -1$ and $\Im_3(\overline{j}) = 0$.

(50) $\Re(\overline{k}) = 0$ and $\Im_1(\overline{k}) = 0$ and $\Im_2(\overline{k}) = 0$ and $\Im_3(\overline{k}) = -1$.

(51) $\overline{i} = -i$.

(52) $\overline{j} = -j$.

(53) $\overline{k} = -k$.

(54) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(55) $\overline{-z} = -\overline{z}$.

(56) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(57) If $\Im_2(z_1) \cdot \Im_3(z_2) \neq \Im_3(z_1) \cdot \Im_2(z_2)$, then $\overline{z_1 \cdot z_2} \neq \overline{z_1} \cdot \overline{z_2}$.

(58) If $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $\overline{z} = z$.

(59) If $\Re(z) = 0$, then $\overline{z} = -z$.

(60) $\Re(z \cdot \overline{z}) = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot \overline{z}) = 0$ and $\Im_2(z \cdot \overline{z}) = 0$ and $\Im_3(z \cdot \overline{z}) = 0$.

(61) $\Re(z+\overline{z}) = 2 \cdot \Re(z)$ and $\Im_1(z+\overline{z}) = 0$ and $\Im_2(z+\overline{z}) = 0$ and $\Im_3(z+\overline{z}) = 0$.

(62) $-z = \langle -\Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(63) $z_1 - z_2 = \langle \Re(z_1) - \Re(z_2), \Im_1(z_1) - \Im_1(z_2), \Im_2(z_1) - \Im_2(z_2), \Im_3(z_1) - \Im_3(z_2) \rangle_{\mathbb{H}}$.

(64) $\Re(z - \overline{z}) = 0$ and $\Im_1(z - \overline{z}) = 2 \cdot \Im_1(z)$ and $\Im_2(z - \overline{z}) = 2 \cdot \Im_2(z)$ and $\Im_3(z - \overline{z}) = 2 \cdot \Im_3(z)$.

Let us consider z. The functor $|z|$ yielding a real number is defined by:

(Def. 27) $|z| = \sqrt{(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}}$.

We now state a number of propositions:

(65) $|0_{\mathbb{H}}| = 0$.

(66) If $|z| = 0$, then $z = 0$.

(67) $0 \leq |z|$.

(68) $|1_{\mathbb{H}}| = 1$.

(69) $|i| = 1$.

(70) $|j| = 1$.

(71) $|k| = 1$.

(72) $|-z| = |z|$.

(73) $|\overline{z}| = |z|$.

(74) $0 \leq (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(75) $\Re(z) \leq |z|$.

(76) $\Im_1(z) \leq |z|$.

(77) $\Im_2(z) \leq |z|$.

(78) $\Im_3(z) \leq |z|$.

(79) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(80) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(81) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(82) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(83) $|z_1 - z_2| = |z_2 - z_1|$.

(84) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(85) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(86) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(87) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(88) $|z \cdot z| = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(89) $|z \cdot z| = |z \cdot \overline{z}|$.

References

[1] Grzegorz Bancerek. Arithmetic of non-negative rational numbers. *To appear in Formalized Mathematics.*

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[10] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.
[11] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[18] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received November 14, 2006