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The Catalan Numbers. Part II[1]

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Summary. In this paper, we define sequence dominated by 0, in which every initial fragment contains more zeroes than ones. If $n \geq 2 \cdot m$ and $n > 0$, then the number of sequences dominated by 0 the length n including m of ones, is given by the formula

$$D(n,m) = \frac{n+1-2\cdot m}{n+1-m} \cdot \binom{n}{m}$$

and satisfies the recurrence relation

$$D(n,m) = D(n-1,m) \ + \sum_{i=0}^{m-1} D(2\cdot i, i) \cdot D(n-2\cdot(i+1), m-(i+1)).$$

Obviously, if $n = 2 \cdot m$, then we obtain the recurrence relation for the Catalan numbers (starting from 0)

$$C_{m+1} = \sum_{i=0}^{m-1} C_{i+1} \cdot C_{m-i}.$$

Using the above recurrence relation we can see that

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \ = 1 + \left(\sum_{i=0}^{\infty} C_{i+1} \cdot x^i\right)^2$$

where $(|x| < \frac{1}{4})$ and hence

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \ = \frac{1-\sqrt{1-4\cdot x}}{2 \cdot x}.$$

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The notation and terminology used here are introduced in the following papers: [2], [23], [7], [25], [19], [27], [5], [28], [9], [1], [26], [21], [6], [3], [14], [12], [16], [13], [20], [15], [8], [22], [11], [10], [18], [24], [17], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: x, D denote sets, i, j, k, l, m, n denote elements of $\mathbb{N}$, p, q denote finite 0-sequences of $\mathbb{N}$, p', q' denote finite 0-sequences, and p_1, q_1 denote finite 0-sequences of D.

Next we state several propositions:

(1) $(p' \frown q') \restriction \operatorname{dom} p' = p'$.

(2) If $n \leq \operatorname{dom} p'$, then $(p' \frown q') \restriction n = p' \restriction n$.

(3) If $n = \operatorname{dom} p' + k$, then $(p' \frown q') \restriction n = p' \frown (q' \restriction k)$.

(4) There exists q' such that $p' = (p' \restriction n) \frown q'$.

(5) There exists q_1 such that $p_1 = (p_1 \restriction n) \frown q_1$.

Let us consider p. We say that p is dominated by 0 if and only if:

(Def. 1) $\operatorname{rng} p \subseteq \{0, 1\}$ and for every k such that $k \leq \operatorname{dom} p$ holds $2 \cdot \sum (p \restriction k) \leq k$.

The following propositions are true:

(6) If p is dominated by 0, then $2 \cdot \sum (p \restriction k) \leq k$.

(7) If p is dominated by 0, then $p(0) = 0$.

Let us consider k, m. Then $k \longmapsto m$ is a finite 0-sequence of $\mathbb{N}$.

One can check that there exists a finite 0-sequence of $\mathbb{N}$ which is empty and dominated by 0 and there exists a finite 0-sequence of $\mathbb{N}$ which is non empty and dominated by 0.

The following propositions are true:

(8) $n \longmapsto 0$ is dominated by 0.

(9) If $n \geq m$, then $(n \longmapsto 0) \frown (m \longmapsto 1)$ is dominated by 0.

(10) If p is dominated by 0, then $p \restriction n$ is dominated by 0.

(11) If p is dominated by 0 and q is dominated by 0, then $p \frown q$ is dominated by 0.

(12) If p is dominated by 0, then $2 \cdot \sum (p \restriction (2 \cdot n + 1)) < 2 \cdot n + 1$.

(13) If p is dominated by 0 and $n \leq \operatorname{len} p - 2 \cdot \sum p$, then $p \frown (n \longmapsto 1)$ is dominated by 0.

(14) If p is dominated by 0 and $n \leq (k + \operatorname{len} p) - 2 \cdot \sum p$, then $(k \longmapsto 0) \frown p \frown (n \longmapsto 1)$ is dominated by 0.

(15) If p is dominated by 0 and $2 \cdot \sum (p \restriction k) = k$, then $k \leq \operatorname{len} p$ and $\operatorname{len}(p \restriction k) = k$.

(16) If p is dominated by 0 and $2 \cdot \sum (p \restriction k) = k$ and $p = (p \restriction k) \frown q$, then q is dominated by 0.

(17) If p is dominated by 0 and $2 \cdot \sum(p \restriction k) = k$ and $k = n + 1$, then $p \restriction k = (p \restriction n) \frown (1 \longmapsto 1)$.

(18) Let given m, p. Suppose $m = \min^*\{n : 2 \cdot \sum(p \restriction n) = n \ \wedge \ n > 0\}$ and $m > 0$ and p is dominated by 0. Then there exists q such that $p \restriction m = (1 \longmapsto 0) \frown q \frown (1 \longmapsto 1)$ and q is dominated by 0.

(19) Let given p. Suppose $\operatorname{rng} p \subseteq \{0, 1\}$ and p is not dominated by 0. Then there exists k such that $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \restriction n) > n\}$ and $2 \cdot k + 1 \leq \operatorname{dom} p$ and $k = \sum(p \restriction (2 \cdot k))$ and $p(2 \cdot k) = 1$.

(20) Let given p, q, k. Suppose $p \restriction (2 \cdot k + \operatorname{len} q) = (k \longmapsto 0) \frown q \frown (k \longmapsto 1)$ and q is dominated by 0 and $2 \cdot \sum q = \operatorname{len} q$ and $k > 0$. Then $\min^*\{n : 2 \cdot \sum(p \restriction n) = n \ \wedge \ n > 0\} = 2 \cdot k + \operatorname{len} q$.

(21) Let given p. Suppose p is dominated by 0 and $\{i : 2 \cdot \sum(p \restriction i) = i \ \wedge \ i > 0\} = \emptyset$ and $\operatorname{len} p > 0$. Then there exists q such that $p = \langle 0 \rangle \frown q$ and q is dominated by 0.

(22) If p is dominated by 0, then $\langle 0 \rangle \frown p$ is dominated by 0 and $\{i : 2 \cdot \sum((\langle 0 \rangle \frown p) \restriction i) = i \ \wedge \ i > 0\} = \emptyset$.

(23) If $\operatorname{rng} p \subseteq \{0, 1\}$ and p is not dominated by 0 and $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \restriction n) > n\}$, then $p \restriction (2 \cdot k)$ is dominated by 0.

2. The Recurrence Relation for the Catalan Numbers

Let n, m be natural numbers. The functor $\operatorname{Domin}_0(n, m)$ yields a subset of $\{0, 1\}^\omega$ and is defined as follows:

(Def. 2) $x \in \operatorname{Domin}_0(n, m)$ iff there exists a finite 0-sequence p of $\mathbb{N}$ such that $p = x$ and p is dominated by 0 and $\operatorname{dom} p = n$ and $\sum p = m$.

Next we state two propositions:

(24) $p \in \operatorname{Domin}_0(n, m)$ iff p is dominated by 0 and $\operatorname{dom} p = n$ and $\sum p = m$.

(25) $\operatorname{Domin}_0(n, m) \subseteq \operatorname{Choose}(n, m, 1, 0)$.

Let us consider n, m. One can check that $\operatorname{Domin}_0(n, m)$ is finite.

One can prove the following propositions:

(26) $\operatorname{Domin}_0(n, m)$ is empty iff $2 \cdot m > n$.

(27) $\operatorname{Domin}_0(n, 0) = \{n \longmapsto 0\}$.

(28) $\operatorname{card} \operatorname{Domin}_0(n, 0) = 1$.

(29) Let given p, n. Suppose $\operatorname{rng} p \subseteq \{0, n\}$. Then there exists q such that $\operatorname{len} p = \operatorname{len} q$ and $\operatorname{rng} q \subseteq \{0, n\}$ and for every k such that $k \leq \operatorname{len} p$ holds $\sum(p \restriction k) + \sum(q \restriction k) = n \cdot k$ and for every k such that $k \in \operatorname{len} p$ holds $q(k) = n - p(k)$.

(30) If $m \leq n$, then $\binom{n}{m} > 0$.

(31) If $2\cdot(m+1) \leq n$, then $\mathrm{card}(\mathrm{Choose}(n, m+1, 1, 0) \setminus \mathrm{Domin}_0(n, m+1)) = \mathrm{card}\,\mathrm{Choose}(n, m, 1, 0)$.

(32) If $2\cdot(m+1) \leq n$, then $\mathrm{card}\,\mathrm{Domin}_0(n, m+1) = \binom{n}{m+1} - \binom{n}{m}$.

(33) If $2\cdot m \leq n$, then $\mathrm{card}\,\mathrm{Domin}_0(n, m) = \frac{(n+1)-2\cdot m}{(n+1)-m} \cdot \binom{n}{m}$.

(34) $\mathrm{card}\,\mathrm{Domin}_0(2+k, 1) = k+1$.

(35) $\mathrm{card}\,\mathrm{Domin}_0(4+k, 2) = \frac{(k+1)\cdot(k+4)}{2}$.

(36) $\mathrm{card}\,\mathrm{Domin}_0(6+k, 3) = \frac{(k+1)\cdot(k+5)\cdot(k+6)}{6}$.

(37) $\mathrm{card}\,\mathrm{Domin}_0(2\cdot n, n) = \frac{\binom{2\cdot n}{n}}{n+1}$.

(38) $\mathrm{card}\,\mathrm{Domin}_0(2\cdot n, n) = \mathrm{Catalan}(n+1)$.

Let us consider D. A functional non empty set is said to be a set of ω-sequences of D if:

(Def. 3) For every x such that $x \in$ it holds x is a finite 0-sequence of D.

Let us consider D. Then D^ω is a set of ω-sequences of D. Let F be a set of ω-sequences of D. We see that the element of F is a finite 0-sequence of D.

In the sequel p_2 denotes an element of $\mathbb{N}^\omega$.

We now state several propositions:

(39) $\overline{\overline{\{p_2 : \mathrm{dom}\, p_2 = 2\cdot n \ \wedge\ p_2 \text{ is dominated by } 0\}}} = \binom{2\cdot n}{n}$.

(40) Let given n, m, k, j, l. Suppose $j = n - 2\cdot(k+1)$ and $l = m - (k+1)$. Then $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n, m) \ \wedge\ 2\cdot(k+1) = \min^*\{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\}\}}} = \mathrm{card}\,\mathrm{Domin}_0(2\cdot k, k)\cdot\mathrm{card}\,\mathrm{Domin}_0(j, l)$.

(41) Let given n, m. Suppose $2\cdot m \leq n$. Then there exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n, m) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} \neq \emptyset\}}} = \sum C_1$ and $\mathrm{dom}\, C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \mathrm{card}\,\mathrm{Domin}_0(2\cdot j, j)\cdot\mathrm{card}\,\mathrm{Domin}_0(n -' 2\cdot(j+1), m -' (j+1))$.

(42) For every n such that $n > 0$ holds $\mathrm{Domin}_0(2\cdot n, n) = \{p_2 : p_2 \in \mathrm{Domin}_0(2\cdot n, n) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} \neq \emptyset\}$.

(43) Let given n. Suppose $n > 0$. Then there exists a finite 0-sequence C_2 of $\mathbb{N}$ such that $\sum C_2 = \mathrm{Catalan}(n+1)$ and $\mathrm{dom}\, C_2 = n$ and for every j such that $j < n$ holds $C_2(j) = \mathrm{Catalan}(j+1)\cdot\mathrm{Catalan}(n -' j)$.

(44) $\overline{\overline{\{p_2 : p_2 \in \mathrm{Domin}_0(n+1, m) \ \wedge\ \{i : 2\cdot\sum(p_2{\restriction}i) = i \ \wedge\ i > 0\} = \emptyset\}}} = \mathrm{card}\,\mathrm{Domin}_0(n, m)$.

(45) Let given n, m. Suppose $2\cdot m \leq n$. Then there exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\mathrm{card}\,\mathrm{Domin}_0(n, m) = \sum C_1 + \mathrm{card}\,\mathrm{Domin}_0(n -' 1, m)$ and $\mathrm{dom}\, C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \mathrm{card}\,\mathrm{Domin}_0(2\cdot j, j)\cdot\mathrm{card}\,\mathrm{Domin}_0(n -' 2\cdot(j+1), m -' (j+1))$.

(46) For all n, k there exists p such that $\sum p = \mathrm{card}\,\mathrm{Domin}_0(2\cdot n + 2 + k, n+1)$ and $\mathrm{dom}\, p = k+1$ and for every i such that $i \leq k$ holds $p(i) =$

card $\mathrm{Domin}_0(2 \cdot n + 1 + i, n)$.

3. Cauchy Product

We use the following convention: s_1, s_2, s_3 denote sequences of real numbers, r denotes a real number, and F_1, F_2, F_3 denote finite 0-sequences of $\mathbb{R}$.

Let us consider F_1. The functor $\sum F_1$ yields a real number and is defined as follows:

(Def. 4) $\sum F_1 = +_{\mathbb{R}} \odot F_1$.

Let us consider F_1, x. Then $F_1(x)$ is a real number.

Let s_1, s_2 be sequences of real numbers. The functor $s_1\,(\#)\,s_2$ yields a sequence of real numbers and is defined by the condition (Def. 5).

(Def. 5) Let k be a natural number. Then there exists a finite 0-sequence F_1 of $\mathbb{R}$ such that $\mathrm{dom}\, F_1 = k + 1$ and for every n such that $n \in k + 1$ holds $F_1(n) = s_1(n) \cdot s_2(k -' n)$ and $\sum F_1 = (s_1\,(\#)\,s_2)(k)$.

Let us notice that the functor $s_1\,(\#)\,s_2$ is commutative.

One can prove the following propositions:

(47) For all F_1, n such that $n \in \mathrm{dom}\, F_1$ holds $\sum(F_1 \restriction n) + F_1(n) = \sum(F_1 \restriction (n + 1))$.

(48) For all F_2, F_3 such that $\mathrm{dom}\, F_2 = \mathrm{dom}\, F_3$ and for every n such that $n \in \mathrm{len}\, F_2$ holds $F_2(n) = F_3(\mathrm{len}\, F_2 -' (1 + n))$ holds $\sum F_2 = \sum F_3$.

(49) For all F_2, F_3 such that $\mathrm{dom}\, F_2 = \mathrm{dom}\, F_3$ and for every n such that $n \in \mathrm{len}\, F_2$ holds $F_2(n) = r \cdot F_3(n)$ holds $\sum F_2 = r \cdot \sum F_3$.

(50) $s_1\,(\#)\,r\,s_2 = r\,(s_1\,(\#)\,s_2)$.

(51) $s_1\,(\#)(s_2 + s_3) = (s_1\,(\#)\,s_2) + (s_1\,(\#)\,s_3)$.

(52) $(s_1\,(\#)\,s_2)(0) = s_1(0) \cdot s_2(0)$.

(53) For all s_1, s_2, n there exists F_1 such that $(\sum_{\alpha=0}^{\kappa}(s_1\,(\#)\,s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum F_1$ and $\mathrm{dom}\, F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(n -' i)$.

(54) Let given s_1, s_2, n. Suppose s_2 is summable. Then there exists F_1 such that $(\sum_{\alpha=0}^{\kappa}(s_1\,(\#)\,s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum s_2 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) - \sum F_1$ and $\mathrm{dom}\, F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot \sum(s_2 \uparrow ((n -' i) + 1))$.

(55) For every F_1 there exists a finite 0-sequence a_1 of $\mathbb{R}$ such that $\mathrm{dom}\, a_1 = \mathrm{dom}\, F_1$ and $|\sum F_1| \leq \sum a_1$ and for every i such that $i \in \mathrm{dom}\, a_1$ holds $a_1(i) = |F_1(i)|$.

(56) For every s_1 such that s_1 is summable there exists r such that $0 < r$ and for every k holds $|\sum(s_1 \uparrow k)| < r$.

(57) For all s_1, n, m such that $n \leq m$ and for every i holds $s_1(i) \geq 0$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(58) For all s_1, s_2 such that s_1 is absolutely summable and s_2 is summable holds $s_1\,(\#)\,s_2$ is summable and $\sum(s_1\,(\#)\,s_2) = \sum s_1 \cdot \sum s_2$.

(59) If $p = F_1$, then $\sum p = \sum F_1$.

4. The Generating Function for the Catalan Numbers

Next we state the proposition

(60) Let given r. Then there exists a sequence C_2 of real numbers such that

(i) for every n holds $C_2(n) = \mathrm{Catalan}(n+1) \cdot r^n$, and

(ii) if $|r| < \frac{1}{4}$, then C_2 is absolutely summable and $\sum C_2 = 1 + r \cdot (\sum C_2)^2$ and $\sum C_2 = \frac{2}{1+\sqrt{1-4\cdot r}}$ and if $r \neq 0$, then $\sum C_2 = \frac{1-\sqrt{1-4\cdot r}}{2\cdot r}$.

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