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Integral of Real-Valued Measurable Function[1]

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Summary. Based on [16], authors formalized the integral of an extended real valued measurable function in [12] before. However, the integral argued in [12] cannot be applied to real-valued functions unconditionally. Therefore, in this article we have formalized the integral of a real-value function.

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The papers [25], [11], [26], [1], [23], [24], [17], [18], [8], [27], [10], [2], [19], [7], [20], [6], [9], [3], [4], [5], [13], [14], [15], [22], [21], and [12] provide the terminology and notation for this paper.

1. The Measurability of Real-Valued Functions

For simplicity, we follow the rules: X denotes a non empty set, Y denotes a set, S denotes a σ-field of subsets of X, F denotes a function from $\mathbb{N}$ into S, f, g denote partial functions from X to $\mathbb{R}$, A, B denote elements of S, r, s denote real numbers, a denotes a real number, and n denotes a natural number.

Let X be a non empty set, let f be a partial function from X to $\mathbb{R}$, and let a be a real number. The functor LE-dom(f, a) yields a subset of X and is defined as follows:

(Def. 1) LE-dom$(f, a) =$ LE-dom$(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

The following three propositions are true:

(1) $|\overline{\mathbb{R}}(f)| = \overline{\mathbb{R}}(|f|)$.

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(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\operatorname{dom} f \in S$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then f is simple function in S.

(3) For every set x holds $x \in \text{LE-dom}(f, a)$ iff $x \in \operatorname{dom} f$ and there exists a real number y such that $y = f(x)$ and $y < a$.

Let us consider X, f, a. The functor $\text{LEQ-dom}(f, a)$ yields a subset of X and is defined as follows:

(Def. 2) $\text{LEQ-dom}(f, a) = \text{LEQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

We now state the proposition

(4) For every set x holds $x \in \text{LEQ-dom}(f, a)$ iff $x \in \operatorname{dom} f$ and there exists a real number y such that $y = f(x)$ and $y \leq a$.

Let us consider X, f, a. The functor $\text{GT-dom}(f, a)$ yielding a subset of X is defined as follows:

(Def. 3) $\text{GT-dom}(f, a) = \text{GT-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

We now state the proposition

(5) For every set x holds $x \in \text{GT-dom}(f, r)$ iff $x \in \operatorname{dom} f$ and there exists a real number y such that $y = f(x)$ and $r < y$.

Let us consider X, f, a. The functor $\text{GTE-dom}(f, a)$ yields a subset of X and is defined as follows:

(Def. 4) $\text{GTE-dom}(f, a) = \text{GTE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

Next we state the proposition

(6) For every set x holds $x \in \text{GTE-dom}(f, r)$ iff $x \in \operatorname{dom} f$ and there exists a real number y such that $y = f(x)$ and $r \leq y$.

Let us consider X, f, a. The functor $\text{EQ-dom}(f, a)$ yielding a subset of X is defined by:

(Def. 5) $\text{EQ-dom}(f, a) = \text{EQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

The following propositions are true:

(7) For every set x holds $x \in \text{EQ-dom}(f, r)$ iff $x \in \operatorname{dom} f$ and there exists a real number y such that $y = f(x)$ and $r = y$.

(8) If for every n holds $F(n) = Y \cap \text{GT-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{GTE-dom}(f, r) = \bigcap \operatorname{rng} F$.

(9) If for every n holds $F(n) = Y \cap \text{LE-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{LEQ-dom}(f, r) = \bigcap \operatorname{rng} F$.

(10) If for every n holds $F(n) = Y \cap \text{LEQ-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{LE-dom}(f, r) = \bigcup \operatorname{rng} F$.

(11) If for every n holds $F(n) = Y \cap \text{GTE-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{GT-dom}(f, r) = \bigcup \operatorname{rng} F$.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\mathbb{R}$, and let A be an element of S. We say that f is measurable on A if and only if:

(Def. 6) $\overline{\mathbb{R}}(f)$ is measurable on A.

The following propositions are true:

(12) f is measurable on A iff for every real number r holds $A \cap \text{LE-dom}(f, r)$ is measurable on S.

(13) Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GTE-dom}(f, r)$ is measurable on S.

(14) f is measurable on A iff for every real number r holds $A \cap \text{LEQ-dom}(f, r)$ is measurable on S.

(15) Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GT-dom}(f, r)$ is measurable on S.

(16) If $B \subseteq A$ and f is measurable on A, then f is measurable on B.

(17) If f is measurable on A and f is measurable on B, then f is measurable on $A \cup B$.

(18) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $A \cap \text{GT-dom}(f, r) \cap \text{LE-dom}(f, s)$ is measurable on S.

(19) If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $A \cap \text{LE-dom}(f, r) \cap \text{GT-dom}(g, r)$ is measurable on S.

(20) $\overline{\mathbb{R}}(r\, f) = r\, \overline{\mathbb{R}}(f)$.

(21) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $r\, f$ is measurable on A.

2. The Measurability of $f + g$ and $f - g$ for Real-Valued Functions f, g

For simplicity, we adopt the following rules: X denotes a non empty set, S denotes a σ-field of subsets of X, f, g denote partial functions from X to $\mathbb{R}$, A denotes an element of S, r denotes a real number, and p denotes a rational number.

Next we state several propositions:

(22) $\overline{\mathbb{R}}(f)$ is finite.

(23) $\overline{\mathbb{R}}(f + g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(g)$ and $\overline{\mathbb{R}}(f - g) = \overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g)$ and $\operatorname{dom} \overline{\mathbb{R}}(f + g) = \operatorname{dom} \overline{\mathbb{R}}(f) \cap \operatorname{dom} \overline{\mathbb{R}}(g)$ and $\operatorname{dom} \overline{\mathbb{R}}(f - g) = \operatorname{dom} \overline{\mathbb{R}}(f) \cap \operatorname{dom} \overline{\mathbb{R}}(g)$ and $\operatorname{dom} \overline{\mathbb{R}}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom} \overline{\mathbb{R}}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(24) For every function F from $\mathbb{Q}$ into S such that for every p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$ holds $A \cap \text{LE-dom}(f + g, r) = \bigcup \operatorname{rng} F$.

(25) Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$.

(26) If f is measurable on A and g is measurable on A, then $f+g$ is measurable on A.

(27) $\overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(-g)$.

(28) $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}((-1)\,f)$ and $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}(-f)$.

(29) If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $f - g$ is measurable on A.

3. Basic Properties of Real-Valued Functions, $\max_+ f$ and $\max_- f$

In the sequel X denotes a non empty set, f denotes a partial function from X to $\mathbb{R}$, and r denotes a real number.

Next we state a number of propositions:

(30) $\max_+(\overline{\mathbb{R}}(f)) = \max_+(f)$ and $\max_-(\overline{\mathbb{R}}(f)) = \max_-(f)$.

(31) For every element x of X holds $0 \leq (\max_+(f))(x)$.

(32) For every element x of X holds $0 \leq (\max_-(f))(x)$.

(33) $\max_-(f) = \max_+(-f)$.

(34) For every set x such that $x \in \operatorname{dom} f$ and $0 < (\max_+(f))(x)$ holds $(\max_-(f))(x) = 0$.

(35) For every set x such that $x \in \operatorname{dom} f$ and $0 < (\max_-(f))(x)$ holds $(\max_+(f))(x) = 0$.

(36) $\operatorname{dom} f = \operatorname{dom}(\max_+(f) - \max_-(f))$ and $\operatorname{dom} f = \operatorname{dom}(\max_+(f) + \max_-(f))$.

(37) For every set x such that $x \in \operatorname{dom} f$ holds $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0$.

(38) For every set x such that $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = f(x)$ holds $(\max_-(f))(x) = 0$.

(39) For every set x such that $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = 0$ holds $(\max_-(f))(x) = -f(x)$.

(40) For every set x such that $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = -f(x)$ holds $(\max_+(f))(x) = 0$.

(41) For every set x such that $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = 0$ holds $(\max_+(f))(x) = f(x)$.

(42) $f = \max_+(f) - \max_-(f)$.

(43) $|r| = |\overline{\mathbb{R}}(r)|$.

(44) $\overline{\mathbb{R}}(|f|) = |\overline{\mathbb{R}}(f)|$.

(45) $|f| = \max_+(f) + \max_-(f)$.

4. The Measurability of $\max_+ f, \max_- f$ and $|f|$

In the sequel X denotes a non empty set, S denotes a σ-field of subsets of X, f denotes a partial function from X to $\mathbb{R}$, and A denotes an element of S.

The following propositions are true:

(46) If f is measurable on A, then $\max_+(f)$ is measurable on A.

(47) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_-(f)$ is measurable on A.

(48) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $|f|$ is measurable on A.

5. The Definition and the Measurability of a Real-Valued Simple Function

For simplicity, we adopt the following rules: X is a non empty set, Y is a set, S is a σ-field of subsets of X, f, g, h are partial functions from X to $\mathbb{R}$, A is an element of S, and r is a real number.

Let us consider X, S, f. We say that f is simple function in S if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a finite sequence F of separated subsets of S such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$, and

(ii) for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

Next we state a number of propositions:

(49) f is simple function in S iff $\overline{\mathbb{R}}(f)$ is simple function in S.

(50) If f is simple function in S, then f is measurable on A.

(51) Let X be a set and f be a partial function from X to $\mathbb{R}$. Then f is non-negative if and only if for every set x holds $0 \leq f(x)$.

(52) Let X be a set and f be a partial function from X to $\mathbb{R}$. If for every set x such that $x \in \operatorname{dom} f$ holds $0 \leq f(x)$, then f is non-negative.

(53) Let X be a set and f be a partial function from X to $\mathbb{R}$. Then f is non-positive if and only if for every set x holds $f(x) \leq 0$.

(54) If for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \leq 0$, then f is non-positive.

(55) If f is non-negative, then $f \restriction Y$ is non-negative.

(56) If f is non-negative and g is non-negative, then $f + g$ is non-negative.

(57) If f is non-negative, then if $0 \leq r$, then $r\, f$ is non-negative and if $r \leq 0$, then $r\, f$ is non-positive.

(58) If for every set x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.

(59) If f is non-negative and g is non-negative and h is non-negative, then $f + g + h$ is non-negative.

(60) For every set x such that $x \in \operatorname{dom}(f + g + h)$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.

(61) $\max_+(f)$ is non-negative and $\max_-(f)$ is non-negative.

(62)(i) $\operatorname{dom}(\max_+(f + g) + \max_-(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(ii) $\operatorname{dom}(\max_-(f + g) + \max_+(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iii) $\operatorname{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iv) $\operatorname{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(v) $\max_+(f + g) + \max_-(f)$ is non-negative, and

(vi) $\max_-(f + g) + \max_+(f)$ is non-negative.

(63) $\max_+(f+g)+\max_-(f)+\max_-(g) = \max_-(f+g)+\max_+(f)+\max_+(g)$.

(64) If $0 \leq r$, then $\max_+(r\, f) = r \max_+(f)$ and $\max_-(r\, f) = r \max_-(f)$.

(65) If $0 \leq r$, then $\max_+((-r)\, f) = r \max_-(f)$ and $\max_-((-r)\, f) = r \max_+(f)$.

(66) $\max_+(f{\restriction}Y) = \max_+(f){\restriction}Y$ and $\max_-(f{\restriction}Y) = \max_-(f){\restriction}Y$.

(67) If $Y \subseteq \operatorname{dom}(f + g)$, then $\operatorname{dom}((f + g){\restriction}Y) = Y$ and $\operatorname{dom}(f{\restriction}Y + g{\restriction}Y) = Y$ and $(f + g){\restriction}Y = f{\restriction}Y + g{\restriction}Y$.

(68) EQ-dom$(f, r) = f^{-1}(\{r\})$.

6. Lemmas for a Real-Valued Measurable Function and a Simple Function

For simplicity, we use the following convention: X is a non empty set, S is a σ-field of subsets of X, f, g are partial functions from X to $\mathbb{R}$, A, B are elements of S, and r, s are real numbers.

We now state a number of propositions:

(69) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $A \cap$ GTE-dom$(f, r) \cap$ LE-dom(f, s) is measurable on S.

(70) If f is simple function in S, then $f{\restriction}A$ is simple function in S.

(71) If f is simple function in S, then $\operatorname{dom} f$ is an element of S.

(72) If f is simple function in S and g is simple function in S, then $f + g$ is simple function in S.

(73) If f is simple function in S, then $r\, f$ is simple function in S.

(74) If for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.

(75) There exists a partial function f from X to $\mathbb{R}$ such that f is simple function in S and $\operatorname{dom} f = A$ and for every set x such that $x \in A$ holds $f(x) = r$.

(76) If f is measurable on B and $A = \operatorname{dom} f \cap B$, then $f \restriction B$ is measurable on A.

(77) If $A \subseteq \operatorname{dom} f$ and f is measurable on A and g is measurable on A, then $\max_+(f+g) + \max_-(f)$ is measurable on A.

(78) If $A \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ and f is measurable on A and g is measurable on A, then $\max_-(f+g) + \max_+(f)$ is measurable on A.

(79) If $\operatorname{dom} f \in S$ and $\operatorname{dom} g \in S$, then $\operatorname{dom}(f+g) \in S$.

(80) If $\operatorname{dom} f = A$, then f is measurable on B iff f is measurable on $A \cap B$.

(81) Given an element A of S such that $\operatorname{dom} f = A$. Let c be a real number and B be an element of S. If f is measurable on B, then $c\, f$ is measurable on B.

7. The Integral of a Real-Valued Function

For simplicity, we follow the rules: X is a non empty set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\mathbb{R}$, r is a real number, and E, A, B are elements of S.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. The functor $\int f \,\mathrm{d}M$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 8) $\int f \,\mathrm{d}M = \int \overline{\mathbb{R}}(f) \,\mathrm{d}M$.

The following propositions are true:

(82) If there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative, then $\int f \,\mathrm{d}M = \int^+ \overline{\mathbb{R}}(f) \,\mathrm{d}M$.

(83) If f is simple function in S and f is non-negative, then $\int f \,\mathrm{d}M = \int^+ \overline{\mathbb{R}}(f) \,\mathrm{d}M$ and $\int f \,\mathrm{d}M = \int' \overline{\mathbb{R}}(f) \,\mathrm{d}M$.

(84) If there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative, then $0 \leq \int f \,\mathrm{d}M$.

(85) Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and A misses B. Then $\int f \restriction (A \cup B) \,\mathrm{d}M = \int f \restriction A \,\mathrm{d}M + \int f \restriction B \,\mathrm{d}M$.

(86) If there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative, then $0 \leq \int f \restriction A \,\mathrm{d}M$.

(87) Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f \restriction A \,\mathrm{d}M \leq \int f \restriction B \,\mathrm{d}M$.

(88) If there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f \restriction A \, \mathrm{d}M = 0$.

(89) If $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f \restriction (E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. We say that f is integrable on M if and only if:

(Def. 9) $\overline{\mathbb{R}}(f)$ is integrable on M.

We now state a number of propositions:

(90) If f is integrable on M, then $-\infty < \int f \, \mathrm{d}M$ and $\int f \, \mathrm{d}M < +\infty$.

(91) If f is integrable on M, then $f \restriction A$ is integrable on M.

(92) If f is integrable on M and A misses B, then $\int f \restriction (A \cup B) \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M + \int f \restriction B \, \mathrm{d}M$.

(93) If f is integrable on M and $B = \operatorname{dom} f \setminus A$, then $f \restriction A$ is integrable on M and $\int f \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M + \int f \restriction B \, \mathrm{d}M$.

(94) Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(95) If f is integrable on M, then $|\int f \, \mathrm{d}M| \leq \int |f| \, \mathrm{d}M$.

(96) Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) $\operatorname{dom} f = \operatorname{dom} g$,

(iii) g is integrable on M, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| \leq g(x)$.

Then f is integrable on M and $\int |f| \, \mathrm{d}M \leq \int g \, \mathrm{d}M$.

(97) If $\operatorname{dom} f \in S$ and $0 \leq r$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$, then $\int f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(98) Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M.

(99) If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(100) If f is integrable on M and g is integrable on M, then $f + g$ is integrable on M.

(101) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f + g \, \mathrm{d}M = \int f \restriction E \, \mathrm{d}M + \int g \restriction E \, \mathrm{d}M$.

(102) If f is integrable on M, then $r \, f$ is integrable on M and $\int r \, f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{R}$, and let B be an element of S. The functor $\int\limits_B f\,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined by:

(Def. 10) $\int\limits_B f\,\mathrm{d}M = \int f{\restriction}B\,\mathrm{d}M$.

Next we state two propositions:

(103) Suppose f is integrable on M and g is integrable on M and $B \subseteq \mathrm{dom}(f + g)$. Then $f + g$ is integrable on M and $\int\limits_B f + g\,\mathrm{d}M = \int\limits_B f\,\mathrm{d}M + \int\limits_B g\,\mathrm{d}M$.

(104) If f is integrable on M and f is measurable on B, then $f{\restriction}B$ is integrable on M and $\int\limits_B r\,f\,\mathrm{d}M = \overline{\mathbb{R}}(r) \cdot \int\limits_B f\,\mathrm{d}M$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[5] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[6] Józef Białas. Some properties of the intervals. *Formalized Mathematics*, 5(**1**):21–26, 1996.
[7] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[12] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006.
[13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.
[14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.
[15] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(**3**):525–529, 2001.
[16] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1987.
[17] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[18] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(**5**):841–845, 1990.
[19] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[20] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 3(**2**):279–288, 1992.
[21] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[22] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[23] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
[24] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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