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Some Special Matrices of Real Elements and Their Properties

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Summary. This article describes definitions of positive matrix, negative matrix, nonpositive matrix, nonnegative matrix, nonzero matrix, module matrix of real elements and their main properties, and we also give the basic inequalities in matrices of real elements.

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The terminology and notation used here are introduced in the following articles: [2], [9], [3], [12], [1], [5], [8], [4], [7], [11], [6], and [10].

1. Some Special Matrices of Real Elements

We use the following convention: a, b are elements of $\mathbb{R}$, i, j, n are natural numbers, and M, M_1, M_2, M_3, M_4 are matrices over $\mathbb{R}$ of dimension n.

Let M be a matrix over $\mathbb{R}$. We say that M is positive if and only if:

(Def. 1) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} > 0$.

We say that M is negative if and only if:

(Def. 2) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} < 0$.

We say that M is nonpositive if and only if:

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(Def. 3) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} \leq 0$.

We say that M is nonnegative if and only if:

(Def. 4) For all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} \geq 0$.

Let M_1, M_2 be matrices over $\mathbb{R}$. The predicate $M_1 \sqsubseteq M_2$ is defined as follows:

(Def. 5) For all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1)_{i,j} < (M_2)_{i,j}$.

We say that M_1 is less or equal with M_2 if and only if:

(Def. 6) For all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1)_{i,j} \leq (M_2)_{i,j}$.

Let M be a matrix over $\mathbb{R}$. The functor $|{:}M{:}|$ yielding a matrix over $\mathbb{R}$ is defined by:

(Def. 7) $\mathrm{len}\,|{:}M{:}| = \mathrm{len}\,M$ and $\mathrm{width}\,|{:}M{:}| = \mathrm{width}\,M$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $|{:}M{:}|_{i,j} = |M_{i,j}|$.

Let us consider n and let us consider M. Then $-M$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n and let us consider M_1, M_2. Then $M_1 + M_2$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n and let us consider M_1, M_2. Then $M_1 - M_2$ is a matrix over $\mathbb{R}$ of dimension n.

Let us consider n, let a be an element of $\mathbb{R}$, and let us consider M. Then $a \cdot M$ is a matrix over $\mathbb{R}$ of dimension n.

Let us observe that there exists a matrix over $\mathbb{R}$ which is positive and nonnegative and there exists a matrix over $\mathbb{R}$ which is negative and nonpositive.

Let M be a positive matrix over $\mathbb{R}$. One can check that M^{T} is positive.

Let M be a negative matrix over $\mathbb{R}$. Note that M^{T} is negative.

Let M be a nonpositive matrix over $\mathbb{R}$. One can verify that M^{T} is nonpositive.

Let M be a nonnegative matrix over $\mathbb{R}$. Observe that M^{T} is nonnegative.

Let us consider n. Observe that $\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}^{n \times n}$ is positive and nonnegative and $\begin{pmatrix} -1 & \ldots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \ldots & -1 \end{pmatrix}^{n \times n}$ is negative and nonpositive.

Let us consider n. One can verify that there exists a matrix over $\mathbb{R}$ of dimension n which is positive and nonnegative and there exists a matrix over $\mathbb{R}$ of dimension n which is negative and nonpositive.

We now state a number of propositions:

(1) For every element x_1 of $\mathbb{R}_{\mathrm{F}}$ and for every real number x_2 such that $x_1 = x_2$ holds $-x_1 = -x_2$.

(2) For every matrix M over $\mathbb{R}$ such that $\langle i, j\rangle \in$ the indices of M holds $(-M)_{i,j} = -M_{i,j}$.

(3) For all matrices M_1, M_2 over $\mathbb{R}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\langle i, j\rangle \in$ the indices of M_1 holds $(M_1 - M_2)_{i,j} = (M_1)_{i,j} - (M_2)_{i,j}$.

(4) For every matrix M over $\mathbb{R}$ such that $\operatorname{len}(a \cdot M) = \operatorname{len} M$ and $\operatorname{width}(a \cdot M) = \operatorname{width} M$ and $\langle i, j\rangle \in$ the indices of M holds $(a \cdot M)_{i,j} = a \cdot M_{i,j}$.

(5) The indices of $M =$ the indices of $|{:}M{:}|$.

(6) $|{:}a \cdot M{:}| = |a| \cdot |{:}M{:}|$.

(7) If M is negative, then $-M$ is positive.

(8) If M_1 is positive and M_2 is positive, then $M_1 + M_2$ is positive.

(9) If $-M_2 \sqsubseteq M_1$, then $M_1 + M_2$ is positive.

(10) If M_1 is nonnegative and M_2 is positive, then $M_1 + M_2$ is positive.

(11) If M_1 is positive and M_2 is negative and $|{:}M_2{:}| \sqsubseteq |{:}M_1{:}|$, then $M_1 + M_2$ is positive.

(12) If M_1 is positive and M_2 is negative, then $M_1 - M_2$ is positive.

(13) If $M_2 \sqsubseteq M_1$, then $M_1 - M_2$ is positive.

(14) If $a > 0$ and M is positive, then $a \cdot M$ is positive.

(15) If $a < 0$ and M is negative, then $a \cdot M$ is positive.

(16) If M is positive, then $-M$ is negative.

(17) If M_1 is negative and M_2 is negative, then $M_1 + M_2$ is negative.

(18) If $M_1 \sqsubseteq -M_2$, then $M_1 + M_2$ is negative.

(19) If M_1 is positive and M_2 is negative and $|{:}M_1{:}| \sqsubseteq |{:}M_2{:}|$, then $M_1 + M_2$ is negative.

(20) If $M_1 \sqsubseteq M_2$, then $M_1 - M_2$ is negative.

(21) If M_1 is positive and M_2 is negative, then $M_2 - M_1$ is negative.

(22) If $a < 0$ and M is positive, then $a \cdot M$ is negative.

(23) If $a > 0$ and M is negative, then $a \cdot M$ is negative.

(24) If M is nonnegative, then $-M$ is nonpositive.

(25) If M is negative, then M is nonpositive.

(26) If M_1 is nonpositive and M_2 is nonpositive, then $M_1 + M_2$ is nonpositive.

(27) If M_1 is less or equal with $-M_2$, then $M_1 + M_2$ is nonpositive.

(28) If M_1 is less or equal with M_2, then $M_1 - M_2$ is nonpositive.

(29) If $a \leq 0$ and M is positive, then $a \cdot M$ is nonpositive.

(30) If $a \geq 0$ and M is negative, then $a \cdot M$ is nonpositive.

(31) If $a \geq 0$ and M is nonpositive, then $a \cdot M$ is nonpositive.

(32) If $a \leq 0$ and M is nonnegative, then $a \cdot M$ is nonpositive.

(33) $|{:}M{:}|$ is nonnegative.

(34) If M_1 is positive, then M_1 is nonnegative.

(35) If M is nonpositive, then $-M$ is nonnegative.

(36) If M_1 is nonnegative and M_2 is nonnegative, then $M_1 + M_2$ is nonnegative.

(37) If $-M_1$ is less or equal with M_2, then $M_1 + M_2$ is nonnegative.

(38) If M_2 is less or equal with M_1, then $M_1 - M_2$ is nonnegative.

(39) If $a \geq 0$ and M is positive, then $a \cdot M$ is nonnegative.

(40) If $a \leq 0$ and M is negative, then $a \cdot M$ is nonnegative.

(41) If $a \leq 0$ and M is nonpositive, then $a \cdot M$ is nonnegative.

(42) If $a \geq 0$ and M is nonnegative, then $a \cdot M$ is nonnegative.

(43) If $a \geq 0$ and $b \geq 0$ and M_1 is nonnegative and M_2 is nonnegative, then $a \cdot M_1 + b \cdot M_2$ is nonnegative.

2. Some Basic Inequalities in Matrices of Real Elements

Next we state a number of propositions:

(44) If $M_1 \sqsubseteq M_2$, then M_1 is less or equal with M_2.

(45) If $M_1 \sqsubseteq M_2$ and $M_2 \sqsubseteq M_3$, then $M_1 \sqsubseteq M_3$.

(46) If $M_1 \sqsubseteq M_2$ and $M_3 \sqsubseteq M_4$, then $M_1 + M_3 \sqsubseteq M_2 + M_4$.

(47) If $M_1 \sqsubseteq M_2$, then $M_1 + M_3 \sqsubseteq M_2 + M_3$.

(48) If $M_1 \sqsubseteq M_2$, then $M_3 - M_2 \sqsubseteq M_3 - M_1$.

(49) $|{:}M_1 + M_2{:}|$ is less or equal with $|{:}M_1{:}| + |{:}M_2{:}|$.

(50) If M_1 is less or equal with M_2, then $M_1 - M_3$ is less or equal with $M_2 - M_3$.

(51) If $M_1 - M_3$ is less or equal with $M_2 - M_3$, then M_1 is less or equal with M_2.

(52) If M_1 is less or equal with $M_2 - M_3$, then M_3 is less or equal with $M_2 - M_1$.

(53) If $M_1 - M_2$ is less or equal with M_3, then $M_1 - M_3$ is less or equal with M_2.

(54) If $M_1 \sqsubseteq M_2$ and M_3 is less or equal with M_4, then $M_1 - M_4 \sqsubseteq M_2 - M_3$.

(55) If M_1 is less or equal with M_2 and $M_3 \sqsubseteq M_4$, then $M_1 - M_4 \sqsubseteq M_2 - M_3$.

(56) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_1 - M_3$ is less or equal with $M_2 - M_4$.

(57) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_4 - M_2$ is less or equal with $M_3 - M_1$.

(58) If $M_1 - M_2$ is less or equal with $M_3 - M_4$, then $M_4 - M_3$ is less or equal with $M_2 - M_1$.

(59) If $M_1 + M_2$ is less or equal with M_3, then M_1 is less or equal with $M_3 - M_2$.

(60) If $M_1 + M_2$ is less or equal with $M_3 + M_4$, then $M_1 - M_3$ is less or equal with $M_4 - M_2$.

(61) If $M_1 + M_2$ is less or equal with $M_3 - M_4$, then $M_1 + M_4$ is less or equal with $M_3 - M_2$.

(62) If $M_1 - M_2$ is less or equal with $M_3 + M_4$, then $M_1 - M_4$ is less or equal with $M_3 + M_2$.

(63) If M_1 is less or equal with M_2, then $-M_2$ is less or equal with $-M_1$.

(64) If M_1 is less or equal with $-M_2$, then M_2 is less or equal with $-M_1$.

(65) If $-M_2$ is less or equal with M_1, then $-M_1$ is less or equal with M_2.

(66) If M_1 is positive, then $M_2 \sqsubseteq M_2 + M_1$.

(67) If M_1 is negative, then $M_1 + M_2 \sqsubseteq M_2$.

(68) If M_1 is nonnegative, then M_2 is less or equal with $M_1 + M_2$.

(69) If M_1 is nonpositive, then $M_1 + M_2$ is less or equal with M_2.

(70) If M_1 is nonpositive and M_3 is less or equal with M_2, then $M_3 + M_1$ is less or equal with M_2.

(71) If M_1 is nonpositive and $M_3 \sqsubseteq M_2$, then $M_3 + M_1 \sqsubseteq M_2$.

(72) If M_1 is negative and M_3 is less or equal with M_2, then $M_3 + M_1 \sqsubseteq M_2$.

(73) If M_1 is nonnegative and M_2 is less or equal with M_3, then M_2 is less or equal with $M_1 + M_3$.

(74) If M_1 is positive and M_2 is less or equal with M_3, then $M_2 \sqsubseteq M_1 + M_3$.

(75) If M_1 is nonnegative and $M_2 \sqsubseteq M_3$, then $M_2 \sqsubseteq M_1 + M_3$.

(76) If M_1 is nonnegative, then $M_2 - M_1$ is less or equal with M_2.

(77) If M_1 is positive, then $M_2 - M_1 \sqsubseteq M_2$.

(78) If M_1 is nonpositive, then M_2 is less or equal with $M_2 - M_1$.

(79) If M_1 is negative, then $M_2 \sqsubseteq M_2 - M_1$.

(80) If M_1 is less or equal with M_2, then $M_2 - M_1$ is nonnegative.

(81) If M_1 is nonnegative and $M_2 \sqsubseteq M_3$, then $M_2 - M_1 \sqsubseteq M_3$.

(82) If M_1 is nonpositive and M_2 is less or equal with M_3, then M_2 is less or equal with $M_3 - M_1$.

(83) If M_1 is nonpositive and $M_2 \sqsubseteq M_3$, then $M_2 \sqsubseteq M_3 - M_1$.

(84) If M_1 is negative and M_2 is less or equal with M_3, then $M_2 \sqsubseteq M_3 - M_1$.

(85) If $M_1 \sqsubseteq M_2$ and $a > 0$, then $a \cdot M_1 \sqsubseteq a \cdot M_2$.

(86) If $M_1 \sqsubseteq M_2$ and $a \geq 0$, then $a \cdot M_1$ is less or equal with $a \cdot M_2$.

(87) If $M_1 \sqsubseteq M_2$ and $a < 0$, then $a \cdot M_2 \sqsubseteq a \cdot M_1$.

(88) If $M_1 \sqsubseteq M_2$ and $a \leq 0$, then $a \cdot M_2$ is less or equal with $a \cdot M_1$.

(89) If M_1 is less or equal with M_2 and $a \geq 0$, then $a \cdot M_1$ is less or equal with $a \cdot M_2$.

(90) If M_1 is less or equal with M_2 and $a \leq 0$, then $a \cdot M_2$ is less or equal with $a \cdot M_1$.

(91) If $a \geq 0$ and $a \leq b$ and M_1 is nonnegative and less or equal with M_2, then $a \cdot M_1$ is less or equal with $b \cdot M_2$.

(92) If $a \leq 0$ and $b \leq a$ and M_1 is nonpositive and M_2 is less or equal with M_1, then $a \cdot M_1$ is less or equal with $b \cdot M_2$.

(93) If $a < 0$ and $b \leq a$ and M_1 is negative and $M_2 \sqsubseteq M_1$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(94) If $a \geq 0$ and $a < b$ and M_1 is nonnegative and $M_1 \sqsubseteq M_2$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(95) If $a \geq 0$ and $a < b$ and M_1 is positive and less or equal with M_2, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

(96) If $a > 0$ and $a \leq b$ and M_1 is positive and $M_1 \sqsubseteq M_2$, then $a \cdot M_1 \sqsubseteq b \cdot M_2$.

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