FORMALIZED MATHEMATICS
2006, Vol. **14**, No. **3**, Pages 109–114
DOI: 10.2478/v10037-006-0013-0

Several Differentiation Formulas of Special Functions. Part IV

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Summary. In this article, we give several differentiation formulas of special and composite functions including trigonometric function, polynomial function and logarithmic function.

MML identifier: `FDIFF_8`, version: `7.8.03 4.75.958`

The notation and terminology used here are introduced in the following papers: [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8].

For simplicity, we adopt the following convention: x, a, b, c denote real numbers, n denotes a natural number, Z denotes an open subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) If $x \in \mathrm{dom}\,(\text{the function tan})$, then $(\text{the function cos})(x) \neq 0$.

(2) If $x \in \mathrm{dom}\,(\text{the function cot})$, then $(\text{the function sin})(x) \neq 0$.

(3) If $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$, then for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})(x)^n_{\mathbb{Z}} = \frac{f_1(x)^n_{\mathbb{Z}}}{f_2(x)^n_{\mathbb{Z}}}$.

(4) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_2(x) = x - b$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{-a-b}{(x-b)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then $(\text{the function ln}) \cdot \frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x}$.

(6) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then

ISSN 1426–2630

(i) (the function tan) $\cdot f$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot f)'_{\restriction Z}(x) = \frac{a}{(\text{the function cos})(a \cdot x+b)^{\mathbf{2}}}$.

(7) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
(i) (the function cot) $\cdot f$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot f)'_{\restriction Z}(x) = -\frac{a}{(\text{the function sin})(a \cdot x+b)^{\mathbf{2}}}$.

(8) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
(i) (the function tan) $\cdot \frac{1}{f}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}} \cdot (\text{the function cos})(\frac{1}{x})^{\mathbf{2}}}$.

(9) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
(i) (the function cot) $\cdot \frac{1}{f}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}} \cdot (\text{the function sin})(\frac{1}{x})^{\mathbf{2}}}$.

(10) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot (f_1 + c\, f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
(i) (the function tan) $\cdot (f_1 + c\, f_2)$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot (f_1 + c\, f_2))'_{\restriction Z}(x) = \frac{b+2 \cdot c \cdot x}{(\text{the function cos})(a+b \cdot x+c \cdot x^{\mathbf{2}})^{\mathbf{2}}}$.

(11) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (f_1 + c\, f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
(i) (the function cot) $\cdot (f_1 + c\, f_2)$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot (f_1 + c\, f_2))'_{\restriction Z}(x) = -\frac{b+2 \cdot c \cdot x}{(\text{the function sin})(a+b \cdot x+c \cdot x^{\mathbf{2}})^{\mathbf{2}}}$.

(12) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot (\text{the function exp}))$. Then
(i) (the function tan) $\cdot$(the function exp) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})((\text{the function exp})(x))^{\mathbf{2}}}$.

(13) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (\text{the function exp}))$. Then
(i) (the function cot) $\cdot$(the function exp) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = -\frac{(\text{the function exp})(x)}{(\text{the function sin})((\text{the function exp})(x))^{\mathbf{2}}}$.

(14) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot (\text{the function ln}))$. Then
(i) (the function tan) $\cdot$(the function ln) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan) $\cdot$(the function ln))$'_{\restriction Z}(x) = \frac{1}{x \cdot (\text{the function cos})((\text{the function ln})(x))^{\mathbf{2}}}$.

(15) Suppose $Z \subseteq$ dom((the function cot) $\cdot$(the function ln)). Then

(i) (the function cot) $\cdot$(the function ln) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cot) $\cdot$(the function ln))$'_{\restriction Z}(x) = -\frac{1}{x \cdot (\text{the function sin})((\text{the function ln})(x))^{\mathbf{2}}}$.

(16) Suppose $Z \subseteq$ dom((the function exp) $\cdot$(the function tan)). Then

(i) (the function exp) $\cdot$(the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$(the function tan))$'_{\restriction Z}(x) = \frac{(\text{the function exp})((\text{the function tan})(x))}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(17) Suppose $Z \subseteq$ dom((the function exp) $\cdot$(the function cot)). Then

(i) (the function exp) $\cdot$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$(the function cot))$'_{\restriction Z}(x) = -\frac{(\text{the function exp})((\text{the function cot})(x))}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(18) Suppose $Z \subseteq$ dom((the function ln) $\cdot$(the function tan)). Then

(i) (the function ln) $\cdot$(the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) $\cdot$(the function tan))$'_{\restriction Z}(x) = \frac{1}{(\text{the function cos})(x) \cdot (\text{the function sin})(x)}$.

(19) Suppose $Z \subseteq$ dom((the function ln) $\cdot$(the function cot)). Then

(i) (the function ln) $\cdot$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) $\cdot$(the function cot))$'_{\restriction Z}(x) = -\frac{1}{(\text{the function sin})(x) \cdot (\text{the function cos})(x)}$.

(20) Suppose $Z \subseteq \mathrm{dom}((^{n}_{\mathbb{Z}}) \cdot (\text{the function tan}))$ and $1 \leq n$. Then

(i) $(^{n}_{\mathbb{Z}}) \cdot$ (the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((^{n}_{\mathbb{Z}}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = \frac{n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}}}{(\text{the function cos})(x)^{n+1}_{\mathbb{Z}}}$.

(21) Suppose $Z \subseteq \mathrm{dom}((^{n}_{\mathbb{Z}}) \cdot (\text{the function cot}))$ and $1 \leq n$. Then

(i) $(^{n}_{\mathbb{Z}}) \cdot$ (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((^{n}_{\mathbb{Z}}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = -\frac{n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}}}{(\text{the function sin})(x)^{n+1}_{\mathbb{Z}}}$.

(22) Suppose that

(i) $Z \subseteq$ dom((the function tan)$+\frac{1}{\text{the function cos}}$), and

(ii) for every x such that $x \in Z$ holds $1 + (\text{the function sin})(x) \neq 0$ and $1 - (\text{the function sin})(x) \neq 0$.

Then

(iii) (the function tan)$+\frac{1}{\text{the function cos}}$ is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds ((the function tan)$+\frac{1}{\text{the function cos}}$)$'_{\restriction Z}(x) = \frac{1}{1 - (\text{the function sin})(x)}$.

(23) Suppose that
(i) $Z \subseteq \mathrm{dom}((\text{the function tan}) - \frac{1}{\text{the function cos}})$, and
(ii) for every x such that $x \in Z$ holds $1 - (\text{the function sin})(x) \neq 0$ and $1 + (\text{the function sin})(x) \neq 0$.
Then
(iii) $(\text{the function tan}) - \frac{1}{\text{the function cos}}$ is differentiable on Z, and
(iv) for every x such that $x \in Z$ holds $((\text{the function tan}) - \frac{1}{\text{the function cos}})'_{\restriction Z}(x) = \frac{1}{1 + (\text{the function sin})(x)}$.

(24) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) - \mathrm{id}_Z)$. Then
(i) $(\text{the function tan}) - \mathrm{id}_Z$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) - \mathrm{id}_Z)'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)^{\mathbf{2}}}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(25) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} - \mathrm{id}_Z)$. Then
(i) $-\text{the function cot} - \mathrm{id}_Z$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(-\text{the function cot} - \mathrm{id}_Z)'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)^{\mathbf{2}}}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(26) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{a}\,((\text{the function tan}) \cdot f) - \mathrm{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
(i) $\frac{1}{a}\,((\text{the function tan}) \cdot f) - \mathrm{id}_Z$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(\frac{1}{a}\,((\text{the function tan}) \cdot f) - \mathrm{id}_Z)'_{\restriction Z}(x) = \frac{(\text{the function sin})(a \cdot x)^{\mathbf{2}}}{(\text{the function cos})(a \cdot x)^{\mathbf{2}}}$.

(27) Suppose $Z \subseteq \mathrm{dom}((-\frac{1}{a})\,((\text{the function cot}) \cdot f) - \mathrm{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
(i) $(-\frac{1}{a})\,((\text{the function cot}) \cdot f) - \mathrm{id}_Z$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((-\frac{1}{a})\,((\text{the function cot}) \cdot f) - \mathrm{id}_Z)'_{\restriction Z}(x) = \frac{(\text{the function cos})(a \cdot x)^{\mathbf{2}}}{(\text{the function sin})(a \cdot x)^{\mathbf{2}}}$.

(28) Suppose $Z \subseteq \mathrm{dom}(f\,(\text{the function tan}))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
(i) $f\,(\text{the function tan})$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(f\,(\text{the function tan}))'_{\restriction Z}(x) = \frac{a \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{a \cdot x + b}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(29) Suppose $Z \subseteq \mathrm{dom}(f\,(\text{the function cot}))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
(i) $f\,(\text{the function cot})$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(f\,(\text{the function cot}))'_{\restriction Z}(x) = \frac{a \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{a \cdot x + b}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(30) Suppose $Z \subseteq \mathrm{dom}((\text{the function exp})\ (\text{the function tan}))$. Then
(i) $(\text{the function exp})\ (\text{the function tan})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) (the function tan))$'_{\restriction Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(31) Suppose $Z \subseteq$ dom((the function exp) (the function cot)). Then

(i) (the function exp) (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) (the function cot))$'_{\restriction Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(32) Suppose $Z \subseteq$ dom((the function ln) (the function tan)). Then

(i) (the function ln) (the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) (the function tan))$'_{\restriction Z}(x) = \frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x} + \frac{(\text{the function ln})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(33) Suppose $Z \subseteq$ dom((the function ln) (the function cot)). Then

(i) (the function ln) (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) (the function cot))$'_{\restriction Z}(x) = \frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x} - \frac{(\text{the function ln})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(34) Suppose $Z \subseteq \text{dom}(\frac{1}{f}$ (the function tan)) and for every x such that $x \in Z$ holds $f(x) = x$. Then

(i) $\frac{1}{f}$ (the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($\frac{1}{f}$ (the function tan))$'_{\restriction Z}(x) = -\frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x^{\mathbf{2}}} + \frac{\frac{1}{x}}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(35) Suppose $Z \subseteq \text{dom}(\frac{1}{f}$ (the function cot)) and for every x such that $x \in Z$ holds $f(x) = x$. Then

(i) $\frac{1}{f}$ (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($\frac{1}{f}$ (the function cot))$'_{\restriction Z}(x) = -\frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x^{\mathbf{2}}} - \frac{\frac{1}{x}}{(\text{the function sin})(x)^{\mathbf{2}}}$.

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Received September 29, 2006