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The Definition of Finite Sequences and Matrices of Probability, and Addition of Matrices of Real Elements

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Summary. In this article, we first define finite sequences of probability distribution and matrices of joint probability and conditional probability. We discuss also the concept of marginal probability. Further, we describe some theorems of matrices of real elements including quadratic form.

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The papers [20], [23], [2], [19], [24], [6], [12], [8], [21], [4], [1], [22], [18], [14], [7], [9], [25], [3], [5], [15], [16], [17], [11], [13], and [10] provide the terminology and notation for this paper.

For simplicity, we use the following convention: D denotes a non empty set, i, j, k denote elements of $\mathbb{N}$, n, m denote natural numbers, and e denotes a finite sequence of elements of $\mathbb{R}$.

Let d be a set, let g be a finite sequence of elements of d^*, and let n be a natural number. Then $g(n)$ is a finite sequence of elements of d.

Let x be a real number. Then $\langle x \rangle$ is a finite sequence of elements of $\mathbb{R}$.

Next we state a number of propositions:

(1) Let a be an element of D, m be a non empty natural number, and g be a finite sequence of elements of D. Then $\operatorname{len} g = m$ and for every natural number i such that $i \in \operatorname{dom} g$ holds $g(i) = a$ if and only if $g = m \mapsto a$.

(2) Let a, b be elements of D. Then there exists a finite sequence g of elements of D such that $\operatorname{len} g = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if $i \in \operatorname{Seg} k$, then $g(i) = a$ and if $i \notin \operatorname{Seg} k$, then $g(i) = b$.

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(3) Suppose that for every natural number i such that $i \in \operatorname{dom} e$ holds $0 \leq e(i)$. Let f be a sequence of real numbers. Suppose $f(1) = e(1)$ and for every natural number n such that $0 \neq n$ and $n < \operatorname{len} e$ holds $f(n+1) = f(n) + e(n+1)$. Let n, m be natural numbers. If $n \in \operatorname{dom} e$ and $m \in \operatorname{dom} e$ and $n \leq m$, then $f(n) \leq f(m)$.

(4) Suppose $\operatorname{len} e \geq 1$ and for every natural number i such that $i \in \operatorname{dom} e$ holds $0 \leq e(i)$. Let f be a sequence of real numbers. Suppose $f(1) = e(1)$ and for every natural number n such that $0 \neq n$ and $n < \operatorname{len} e$ holds $f(n+1) = f(n) + e(n+1)$. Let n be a natural number. If $n \in \operatorname{dom} e$, then $e(n) \leq f(n)$.

(5) Suppose that for every natural number i such that $i \in \operatorname{dom} e$ holds $0 \leq e(i)$. Let k be a natural number. If $k \in \operatorname{dom} e$, then $e(k) \leq \sum e$.

(6) Let r_1, r_2 be real numbers, k be a natural number, and s_2 be a sequence of real numbers. Then there exists a sequence s_1 of real numbers such that $s_1(0) = r_1$ and for every n holds if $n \neq 0$ and $n \leq k$, then $s_1(n) = s_2(n)$ and if $n > k$, then $s_1(n) = r_2$.

(7) Let F be a finite sequence of elements of $\mathbb{R}$. Then there exists a sequence f of real numbers such that $f(0) = 0$ and for every natural number i such that $i < \operatorname{len} F$ holds $f(i+1) = f(i) + F(i+1)$ and $\sum F = f(\operatorname{len} F)$.

(8) Let D be a set and e_1 be a finite sequence of elements of D. Then $n \mapsto e_1$ is a finite sequence of elements of D^*.

(9) Let D be a set and e_1, e_2 be finite sequences of elements of D. Then there exists a finite sequence e of elements of D^* such that $\operatorname{len} e = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if $i \in \operatorname{Seg} k$, then $e(i) = e_1$ and if $i \notin \operatorname{Seg} k$, then $e(i) = e_2$.

(10) Let D be a set and s be a finite sequence. Then s is a matrix over D if and only if there exists n such that for every i such that $i \in \operatorname{dom} s$ there exists a finite sequence p of elements of D such that $s(i) = p$ and $\operatorname{len} p = n$.

(11) Let D be a set and e be a finite sequence of elements of D^*. Then there exists n such that for every i such that $i \in \operatorname{dom} e$ holds $\operatorname{len} e(i) = n$ if and only if e is a matrix over D.

(12) For every tabular finite sequence M holds $\langle i, j\rangle \in$ the indices of M iff $i \in \operatorname{Seg} \operatorname{len} M$ and $j \in \operatorname{Seg} \operatorname{width} M$.

(13) Let D be a non empty set and M be a matrix over D. Then $\langle i, j\rangle \in$ the indices of M if and only if $i \in \operatorname{dom} M$ and $j \in \operatorname{dom} M(i)$.

(14) For every non empty set D and for every matrix M over D such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = M(i)(j)$.

(15) Let D be a non empty set and M be a matrix over D. Then $\langle i, j\rangle \in$ the indices of M if and only if $i \in \operatorname{dom}(M_{\square,j})$ and $j \in \operatorname{dom} \operatorname{Line}(M, i)$.

(16) Let D_1, D_2 be non empty sets, M_1 be a matrix over D_1, and M_2 be a

matrix over D_2. If $M_1 = M_2$, then for every i such that $i \in \operatorname{dom} M_1$ holds $\operatorname{Line}(M_1, i) = \operatorname{Line}(M_2, i)$.

(17) Let D_1, D_2 be non empty sets, M_1 be a matrix over D_1, and M_2 be a matrix over D_2. If $M_1 = M_2$, then for every j such that $j \in \operatorname{Seg} \operatorname{width} M_1$ holds $(M_1)_{\square,j} = (M_2)_{\square,j}$.

(18) Let e_1 be a finite sequence of elements of D. If $\operatorname{len} e_1 = m$, then $n \mapsto e_1$ is a matrix over D of dimension $n \times m$.

(19) Let e_1, e_2 be finite sequences of elements of D. Suppose $\operatorname{len} e_1 = m$ and $\operatorname{len} e_2 = m$. Then there exists a matrix M over D of dimension $n \times m$ such that for every natural number i holds

(i) if $i \in \operatorname{Seg} k$, then $M(i) = e_1$, and

(ii) if $i \notin \operatorname{Seg} k$, then $M(i) = e_2$.

Let e be a finite sequence of elements of $\mathbb{R}^*$. The functor $\sum e$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 1) $\operatorname{len} \sum e = \operatorname{len} e$ and for every k such that $k \in \operatorname{dom} \sum e$ holds $(\sum e)(k) = \sum e(k)$.

Let m be a matrix over $\mathbb{R}$. We introduce $\operatorname{LineSum} m$ as a synonym of $\sum m$.

We now state the proposition

(20) For every matrix m over $\mathbb{R}$ holds $\operatorname{len} \sum m = \operatorname{len} m$ and for every i such that $i \in \operatorname{Seg} \operatorname{len} m$ holds $(\sum m)(i) = \sum \operatorname{Line}(m, i)$.

Let m be a matrix over $\mathbb{R}$. The functor $\operatorname{ColSum} m$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 2) $\operatorname{len} \operatorname{ColSum} m = \operatorname{width} m$ and for every j such that $j \in \operatorname{Seg} \operatorname{width} m$ holds $(\operatorname{ColSum} m)(j) = \sum (m_{\square,j})$.

We now state two propositions:

(21) For every matrix M over $\mathbb{R}$ such that $\operatorname{width} M > 0$ holds $\operatorname{LineSum} M = \operatorname{ColSum}(M^{\mathrm{T}})$.

(22) For every matrix M over $\mathbb{R}$ holds $\operatorname{ColSum} M = \operatorname{LineSum}(M^{\mathrm{T}})$.

Let M be a matrix over $\mathbb{R}$. The functor $\operatorname{SumAll} M$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 3) $\operatorname{SumAll} M = \sum \sum M$.

The following propositions are true:

(23) For every matrix M over $\mathbb{R}$ such that $\operatorname{len} M = 0$ holds $\operatorname{SumAll} M = 0$.

(24) For every matrix M over $\mathbb{R}$ of dimension $m \times 0$ holds $\operatorname{SumAll} M = 0$.

(25) Let M_1 be a matrix over $\mathbb{R}$ of dimension $n \times k$ and M_2 be a matrix over $\mathbb{R}$ of dimension $m \times k$. Then $\sum (M_1 \frown M_2) = (\sum M_1) \frown \sum M_2$.

(26) For all matrices M_1, M_2 over $\mathbb{R}$ holds $\sum M_1 + \sum M_2 = \sum (M_1 \frown M_2)$.

(27) For all matrices M_1, M_2 over $\mathbb{R}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ holds $\operatorname{SumAll} M_1 + \operatorname{SumAll} M_2 = \operatorname{SumAll}(M_1 \frown M_2)$.

(28) For every matrix M over $\mathbb{R}$ holds $\mathrm{SumAll}\, M = \mathrm{SumAll}(M^{\mathrm{T}})$.

(29) For every matrix M over $\mathbb{R}$ holds $\mathrm{SumAll}\, M = \sum \mathrm{ColSum}\, M$.

(30) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $\mathrm{len}(x \bullet y) = \mathrm{len}\, x$.

(31) For every i and for every element R of $\mathbb{R}^i$ holds $i \mapsto 1 \bullet R = R$.

(32) For every finite sequence x of elements of $\mathbb{R}$ holds $\mathrm{len}\, x \mapsto 1 \bullet x = x$.

(33) Let x, y be finite sequences of elements of $\mathbb{R}$. Suppose for every i such that $i \in \mathrm{dom}\, x$ holds $x(i) \geq 0$ and for every i such that $i \in \mathrm{dom}\, y$ holds $y(i) \geq 0$. Let given k. If $k \in \mathrm{dom}(x \bullet y)$, then $(x \bullet y)(k) \geq 0$.

(34) Let given i, e_1, e_2 be elements of $\mathbb{R}^i$, and f_1, f_2 be elements of (the carrier of $\mathbb{R}_{\mathrm{F}})^i$. If $e_1 = f_1$ and $e_2 = f_2$, then $e_1 \bullet e_2 = f_1 \bullet f_2$.

(35) Let e_1, e_2 be finite sequences of elements of $\mathbb{R}$ and f_1, f_2 be finite sequences of elements of $\mathbb{R}_{\mathrm{F}}$. If $\mathrm{len}\, e_1 = \mathrm{len}\, e_2$ and $e_1 = f_1$ and $e_2 = f_2$, then $e_1 \bullet e_2 = f_1 \bullet f_2$.

(36) Let e be a finite sequence of elements of $\mathbb{R}$ and f be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. If $e = f$, then $\sum e = \sum f$.

Let e_1, e_2 be finite sequences of elements of $\mathbb{R}$. We introduce $e_1 \cdot e_2$ as a synonym of $|(e_1, e_2)|$.

We now state several propositions:

(37) Let given i, e_1, e_2 be elements of $\mathbb{R}^i$, and f_1, f_2 be elements of (the carrier of $\mathbb{R}_{\mathrm{F}})^i$. If $e_1 = f_1$ and $e_2 = f_2$, then $e_1 \cdot e_2 = f_1 \cdot f_2$.

(38) Let e_1, e_2 be finite sequences of elements of $\mathbb{R}$ and f_1, f_2 be finite sequences of elements of $\mathbb{R}_{\mathrm{F}}$. If $\mathrm{len}\, e_1 = \mathrm{len}\, e_2$ and $e_1 = f_1$ and $e_2 = f_2$, then $e_1 \cdot e_2 = f_1 \cdot f_2$.

(39) Let M, M_1, M_2 be matrices over $\mathbb{R}$. Suppose $\mathrm{width}\, M_1 = \mathrm{len}\, M_2$. Then $M = M_1 \cdot M_2$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, M = \mathrm{len}\, M_1$,

(ii) $\mathrm{width}\, M = \mathrm{width}\, M_2$, and

(iii) for all i, j such that $\langle i, j \rangle \in$ the indices of M holds $M_{i,j} = \mathrm{Line}(M_1, i) \cdot (M_2)_{\Box, j}$.

(40) Let M be a matrix over $\mathbb{R}$ and p be a finite sequence of elements of $\mathbb{R}$. If $\mathrm{len}\, M = \mathrm{len}\, p$, then for every i such that $i \in \mathrm{Seg}\, \mathrm{len}(p \cdot M)$ holds $(p \cdot M)(i) = p \cdot M_{\Box, i}$.

(41) Let M be a matrix over $\mathbb{R}$ and p be a finite sequence of elements of $\mathbb{R}$. If $\mathrm{width}\, M = \mathrm{len}\, p$ and $\mathrm{width}\, M > 0$, then for every i such that $i \in \mathrm{Seg}\, \mathrm{len}(M \cdot p)$ holds $(M \cdot p)(i) = \mathrm{Line}(M, i) \cdot p$.

(42) Let M, M_1, M_2 be matrices over $\mathbb{R}$. Suppose $\mathrm{width}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 > 0$ and $\mathrm{width}\, M_2 > 0$. Then $M = M_1 \cdot M_2$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, M = \mathrm{len}\, M_1$,

(ii) $\operatorname{width} M = \operatorname{width} M_2$, and

(iii) for every i such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $\operatorname{Line}(M, i) = \operatorname{Line}(M_1, i) \cdot M_2$.

Let n, m, k be non empty natural numbers, let M_1 be a matrix over $\mathbb{R}$ of dimension $n \times k$, and let M_2 be a matrix over $\mathbb{R}$ of dimension $k \times m$. Note that $M_1 \cdot M_2$

let x, y be finite sequences of elements of $\mathbb{R}$ and let M be a matrix over $\mathbb{R}$. Let us assume that $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$. The functor $\operatorname{QuadraticForm}(x, M, y)$ yields a matrix over $\mathbb{R}$ and is defined by the conditions (Def. 4).

(Def. 4)(i) $\operatorname{len} \operatorname{QuadraticForm}(x, M, y) = \operatorname{len} x$,

(ii) $\operatorname{width} \operatorname{QuadraticForm}(x, M, y) = \operatorname{len} y$, and

(iii) for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $(\operatorname{QuadraticForm}(x, M, y))_{i,j} = x(i) \cdot M_{i,j} \cdot y(j)$.

The following propositions are true:

(43) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $(\operatorname{QuadraticForm}(x, M, y))^{\mathrm{T}} = \operatorname{QuadraticForm}(y, M^{\mathrm{T}}, x)$.

(44) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(x, M \cdot y)| = \operatorname{SumAll} \operatorname{QuadraticForm}(x, M, y)$.

(45) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, \operatorname{len} x \mapsto 1)| = \sum x$.

(46) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(x \cdot M, y)| = \operatorname{SumAll} \operatorname{QuadraticForm}(x, M, y)$.

(47) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(x \cdot M, y)| = |(x, M \cdot y)|$.

(48) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} y = \operatorname{len} M$ and $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(M \cdot x, y)| = |(x, M^{\mathrm{T}} \cdot y)|$.

(49) Let x, y be finite sequences of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} y = \operatorname{len} M$ and $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(x, y \cdot M)| = |(x \cdot M^{\mathrm{T}}, y)|$.

(50) Let x be a finite sequence of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x = \operatorname{len} M$ and $x = \operatorname{len} x \mapsto 1$, then for every k such that $k \in \operatorname{Seg} \operatorname{len}(x \cdot M)$ holds $(x \cdot M)(k) = \sum(M_{\square,k})$.

(51) Let x be a finite sequence of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. Suppose $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{width} M > 0$ and $x = \operatorname{len} x \mapsto 1$. Let given k. If $k \in \operatorname{Seg} \operatorname{len}(M \cdot x)$, then $(M \cdot x)(k) = \sum \operatorname{Line}(M, k)$.

(52) Let n be a non empty natural number. Then there exists a finite sequence P of elements of $\mathbb{R}$ such that $\operatorname{len} P = n$ and for every i such that $i \in \operatorname{dom} P$ holds $P(i) \geq 0$ and $\sum P = 1$.

Let p be a finite sequence of elements of $\mathbb{R}$. We say that p is finite probability distribution if and only if:

(Def. 5) For every i such that $i \in \operatorname{dom} p$ holds $p(i) \geq 0$ and $\sum p = 1$.

One can check that there exists a finite sequence of elements of $\mathbb{R}$ which is non empty and finite probability distribution.

One can prove the following propositions:

(53) Let p be a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and given k. If $k \in \operatorname{dom} p$, then $p(k) \leq 1$.

(54) For every non empty yielding matrix M over D holds $1 \leq \operatorname{len} M$ and $1 \leq \operatorname{width} M$.

Let M be a matrix over $\mathbb{R}$. We say that M is nonnegative if and only if:

(Def. 6) For all i, j such that $\langle i, j \rangle \in$ the indices of M holds $M_{i,j} \geq 0$.

Let M be a matrix over $\mathbb{R}$. We say that M is summable-to-1 if and only if:

(Def. 7) $\operatorname{SumAll} M = 1$.

Let M be a matrix over $\mathbb{R}$. We say that M is joint probability if and only if:

(Def. 8) M is nonnegative and summable-to-1.

Let us mention that every matrix over $\mathbb{R}$ which is joint probability is also nonnegative and summable-to-1 and every matrix over $\mathbb{R}$ which is nonnegative and summable-to-1 is also joint probability.

We now state the proposition

(55) Let n, m be non empty natural numbers. Then there exists a matrix M over $\mathbb{R}$ of dimension $n \times m$ such that M is nonnegative and $\operatorname{SumAll} M = 1$.

One can check that there exists a matrix over $\mathbb{R}$ which is non empty yielding and joint probability.

Let n, m be non empty natural numbers, let D be a non empty set, and let M be a matrix over D of dimension $n \times m$. Observe that M^{T}

Next we state two propositions:

(56) Let M be a non empty yielding joint probability matrix over $\mathbb{R}$. Then M^{T} is a non empty yielding joint probability matrix over $\mathbb{R}$.

(57) Let M be a non empty yielding joint probability matrix over $\mathbb{R}$ and given i, j. If $\langle i, j \rangle \in$ the indices of M, then $M_{i,j} \leq 1$.

Let M be a matrix over $\mathbb{R}$. We say that M has lines summable-to-1 if and only if:

(Def. 9) For every k such that $k \in \operatorname{dom} M$ holds $\sum M(k) = 1$.

The following proposition is true

(58) For all non empty natural numbers n, m holds there exists a matrix over $\mathbb{R}$ of dimension $n \times m$ which is nonnegative and has lines summable-to-1.

Let M be a matrix over $\mathbb{R}$. We say that M is conditional probability if and only if:

(Def. 10) M is nonnegative and has lines summable-to-1.

Let us observe that every matrix over $\mathbb{R}$ which is conditional probability is also nonnegative and has lines summable-to-1 and every matrix over $\mathbb{R}$ which is nonnegative and has lines summable-to-1 is also conditional probability.

Let us mention that there exists a matrix over $\mathbb{R}$ which is non empty yielding and conditional probability.

Next we state three propositions:

(59) Let M be a non empty yielding conditional probability matrix over $\mathbb{R}$ and given i, j. If $\langle i, j \rangle \in$ the indices of M, then $M_{i,j} \leq 1$.

(60) Let M be a non empty yielding matrix over $\mathbb{R}$. Then the following statements are equivalent

(i) M is a non empty yielding conditional probability matrix over $\mathbb{R}$,

(ii) for every i such that $i \in \operatorname{dom} M$ holds $\operatorname{Line}(M, i)$ is a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$.

(61) For every non empty yielding matrix M over $\mathbb{R}$ with lines summable-to-1 holds $\operatorname{SumAll} M = \operatorname{len} M$.

Let M be a matrix over $\mathbb{R}$. We introduce the row marginal M as a synonym of $\operatorname{LineSum} M$. We introduce the column marginal M as a synonym of $\operatorname{ColSum} M$.

Let M be a non empty yielding joint probability matrix over $\mathbb{R}$. Note that the row marginal M is non empty and finite probability distribution and the column marginal M is non empty and finite probability distribution.

Let M be a non empty yielding matrix over $\mathbb{R}$. Observe that M^{T} is non empty yielding.

Let M be a non empty yielding joint probability matrix over $\mathbb{R}$. Note that M^{T} is joint probability.

The following propositions are true:

(62) Let p be a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and P be a non empty yielding conditional probability matrix over $\mathbb{R}$. Suppose $\operatorname{len} p = \operatorname{len} P$. Then $p \cdot P$ is a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and $\operatorname{len}(p \cdot P) = \operatorname{width} P$.

(63) Let P_1, P_2 be non empty yielding conditional probability matrices over $\mathbb{R}$. Suppose $\operatorname{width} P_1 = \operatorname{len} P_2$. Then $P_1 \cdot P_2$ is a non empty yielding conditional probability matrix over $\mathbb{R}$ and $\operatorname{len}(P_1 \cdot P_2) = \operatorname{len} P_1$ and $\operatorname{width}(P_1 \cdot P_2) = \operatorname{width} P_2$.

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