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Integral of Measurable Function[1]

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Summary. In this paper we construct integral of measurable function.

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The terminology and notation used here are introduced in the following articles: [29], [12], [32], [1], [27], [18], [33], [9], [2], [34], [13], [11], [10], [28], [31], [20], [30], [3], [4], [5], [14], [7], [17], [15], [16], [26], [8], [19], [21], [24], [23], [6], [22], and [25].

1. Lemmas for Extended Real Numbers

One can prove the following propositions:

(1) For all extended real numbers x, y holds $|x-y| = |y-x|$.

(2) For all extended real numbers x, y holds $y - x \leq |x-y|$.

(3) Let x, y be extended real numbers and e be a real number. Suppose $|x-y| < e$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$. Then $x \neq +\infty$ and $x \neq -\infty$ and $y \neq +\infty$ and $y \neq -\infty$.

(4) For all extended real numbers x, y such that for every real number e such that $0 < e$ holds $x < y + \overline{\mathbb{R}}(e)$ holds $x \leq y$.

(5) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x + t < y + t$.

(6) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x - t < y - t$.

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(7) For all real numbers a, b holds $\overline{\mathbb{R}}(a) + \overline{\mathbb{R}}(b) = a + b$ and $-\overline{\mathbb{R}}(a) = -a$.

(8) Let n be a natural number and p be an extended real number. Suppose $0 \leq p$ and $p < n$. Then there exists a natural number k such that $1 \leq k$ and $k \leq 2^n \cdot n$ and $\frac{k-1}{2^n} \leq p$ and $p < \frac{k}{2^n}$.

(9) Let n, k be natural numbers and p be an extended real number. If $1 \leq k$ and $k \leq 2^n \cdot n$ and $n \leq p$ and $\frac{k-1}{2^n} \leq p$, then $\frac{k}{2^n} \leq p$.

(10) For all extended real numbers x, y, w, z such that $-\infty < w$ holds if $x < y$ and $w < z$, then $x + w < y + z$.

(11) For all extended real numbers x, y, k such that $0 \leq k$ holds $k \cdot \max(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \min(x, y) = \min(k \cdot x, k \cdot y)$.

(12) For all extended real numbers x, y, k such that $k \leq 0$ holds $k \cdot \min(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \max(x, y) = \min(k \cdot x, k \cdot y)$.

(13) For all extended real numbers x, y, z such that $0 \leq x$ and $0 \leq z$ and $z + x \leq y$ holds $z \leq y$.

2. Lemmas for Partial Function of Non-empty Set, Extended Real Numbers

Let I_1 be a set. We say that I_1 is non-positive if and only if:

(Def. 1) For every extended real number x such that $x \in I_1$ holds $x \leq 0$.

Let R be a binary relation. We say that R is non-positive if and only if:

(Def. 2) $\operatorname{rng} R$ is non-positive.

The following propositions are true:

(14) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. Then F is non-positive if and only if for every set n holds $F(n) \leq 0_{\overline{\mathbb{R}}}$.

(15) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. If for every set n such that $n \in \operatorname{dom} F$ holds $F(n) \leq 0_{\overline{\mathbb{R}}}$, then F is non-positive.

Let R be a binary relation. We say that R is without $-\infty$ if and only if:

(Def. 3) $-\infty \notin \operatorname{rng} R$.

We say that R is without $+\infty$ if and only if:

(Def. 4) $+\infty \notin \operatorname{rng} R$.

Let X be a non empty set and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us observe that f is without $-\infty$ if and only if:

(Def. 5) For every set x holds $-\infty < f(x)$.

Let us observe that f is without $+\infty$ if and only if:

(Def. 6) For every set x holds $f(x) < +\infty$.

Next we state four propositions:

(16) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \operatorname{dom} f$ holds $-\infty < f(x)$ if and only if f is without $-\infty$.

(17) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \operatorname{dom} f$ holds $f(x) < +\infty$ if and only if f is without $+\infty$.

(18) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then f is without $-\infty$.

(19) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-positive, then f is without $+\infty$.

Let X be a non empty set. Note that every partial function from X to $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every partial function from X to $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$.

The following propositions are true:

(20) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then f is without $+\infty$ and without $-\infty$.

(21) Let X be a non empty set, Y be a set, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then $f \restriction Y$ is non-negative.

(22) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(23) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $+\infty$. Then $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(24) Let X be a non empty set, S be a σ-field of subsets of X, f, g be partial functions from X to $\overline{\mathbb{R}}$, F be a function from $\mathbb{Q}$ into S, r be a real number, and A be an element of S. Suppose f is without $-\infty$ and g is without $-\infty$ and for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$. Then $A \cap \text{LE-dom}(f+g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{R}$. The functor $\overline{\mathbb{R}}(f)$ yielding a partial function from X to $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) $\overline{\mathbb{R}}(f) = f$.

Next we state a number of propositions:

(25) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is non-negative and g is non-negative, then $f+g$ is non-negative.

(26) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number such that f is non-negative. Then

(i) if $0 \leq c$, then $c\,f$ is non-negative, and

(ii) if $c \leq 0$, then $c\,f$ is non-positive.

(27) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that for every set x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $g(x) \leq f(x)$ and $-\infty < g(x)$ and $f(x) < +\infty$. Then $f - g$ is non-negative.

(28) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative. Then $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $f + g$ is non-negative.

(29) Let X be a non empty set and f, g, h be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative and h is non-negative. Then $\operatorname{dom}(f + g + h) = \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h$ and $f + g + h$ is non-negative and for every set x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.

(30) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then

(i) $\operatorname{dom}(\max_+(f + g) + \max_-(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(ii) $\operatorname{dom}(\max_-(f + g) + \max_+(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iii) $\operatorname{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iv) $\operatorname{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(v) $\max_+(f + g) + \max_-(f)$ is non-negative, and

(vi) $\max_-(f + g) + \max_+(f)$ is non-negative.

(31) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and without $+\infty$ and g is without $-\infty$ and without $+\infty$. Then $\max_+(f+g)+\max_-(f)+\max_-(g) = \max_-(f+g)+\max_+(f)+\max_+(g)$.

(32) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+(c\,f) = c\,\max_+(f)$ and $\max_-(c\,f) = c\,\max_-(f)$.

(33) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+((-c)\,f) = c\,\max_-(f)$ and $\max_-((-c)\,f) = c\,\max_+(f)$.

(34) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and A be a set. Then $\max_+(f{\restriction}A) = \max_+(f){\restriction}A$ and $\max_-(f{\restriction}A) = \max_-(f){\restriction}A$.

(35) Let X be a non empty set, f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be a set. If $B \subseteq \operatorname{dom}(f + g)$, then $\operatorname{dom}((f + g){\restriction}B) = B$ and $\operatorname{dom}(f{\restriction}B + g{\restriction}B) = B$ and $(f + g){\restriction}B = f{\restriction}B + g{\restriction}B$.

(36) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and a be an extended real number. Then $\text{EQ-dom}(f, a) = f^{-1}(\{a\})$.

3. Lemmas for Measurable Function and Simple Valued Function

The following propositions are true:

(37) Let X be a non empty set, S be a σ-field of subsets of X, f, g be partial functions from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose f is without $-\infty$ and g is without $-\infty$ and f is measurable on A and g is measurable on A. Then $f+g$ is measurable on A.

(38) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $\operatorname{dom} f = \emptyset$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are representation of f,

(ii) $a(1) = 0$,

(iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0 < a(n)$ and $a(n) < +\infty$,

(iv) $\operatorname{dom} x = \operatorname{dom} F$,

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and

(vi) $\sum x = 0$.

(39) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S, and r, s be real numbers. Suppose f is measurable on A and $A \subseteq \operatorname{dom} f$. Then $A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r)) \cap \text{LE-dom}(f, \overline{\mathbb{R}}(s))$ is measurable on S.

(40) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. If f is simple function in S, then $f \restriction A$ is simple function in S.

(41) Let X be a non empty set, S be a σ-field of subsets of X, A be an element of S, F be a finite sequence of separated subsets of S, and G be a finite sequence. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $G(n) = F(n) \cap A$. Then G is a finite sequence of separated subsets of S.

(42) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S, F, G be finite sequences of separated subsets of S, and a be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $G(n) = F(n) \cap A$ and F and a are representation of f. Then G and a are representation of $f \restriction A$.

(43) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S, then $\operatorname{dom} f$ is an element of S.

(44) Let X be a non empty set, S be a σ-field of subsets of X, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S. Then $f+g$ is simple function in S.

(45) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. If f is simple function in S, then $c\,f$ is simple function in S.

(46) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) f is simple function in S,
- (ii) g is simple function in S, and
- (iii) for every set x such that $x \in \operatorname{dom}(f-g)$ holds $g(x) \leq f(x)$.

Then $f-g$ is non-negative.

(47) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and c be an extended real number. Suppose $c \neq +\infty$ and $c \neq -\infty$. Then there exists a partial function f from X to $\overline{\mathbb{R}}$ such that f is simple function in S and $\operatorname{dom} f = A$ and for every set x such that $x \in A$ holds $f(x) = c$.

(48) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and B, B_1 be elements of S. Suppose f is measurable on B and $B_1 = \operatorname{dom} f \cap B$. Then $f \restriction B$ is measurable on B_1.

(49) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \operatorname{dom} f$,
- (ii) f is measurable on A,
- (iii) g is measurable on A,
- (iv) f is without $-\infty$, and
- (v) g is without $-\infty$.

Then $\max_{+}(f+g) + \max_{-}(f)$ is measurable on A.

(50) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \operatorname{dom} f \cap \operatorname{dom} g$,
- (ii) f is measurable on A,
- (iii) g is measurable on A,
- (iv) f is without $-\infty$, and
- (v) g is without $-\infty$.

Then $\max_{-}(f+g) + \max_{+}(f)$ is measurable on A.

(51) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and A be a set. If $A \in S$, then $0 \leq M(A)$.

(52) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
(i) there exists an element E_1 of S such that $E_1 = \operatorname{dom} f$ and f is measurable on E_1,
(ii) there exists an element E_2 of S such that $E_2 = \operatorname{dom} g$ and g is measurable on E_2,
(iii) $f^{-1}(\{+\infty\}) \in S$,
(iv) $f^{-1}(\{-\infty\}) \in S$,
(v) $g^{-1}(\{+\infty\}) \in S$, and
(vi) $g^{-1}(\{-\infty\}) \in S$.
Then $\operatorname{dom}(f+g) \in S$.

(53) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
(i) there exists an element E_1 of S such that $E_1 = \operatorname{dom} f$ and f is measurable on E_1, and
(ii) there exists an element E_2 of S such that $E_2 = \operatorname{dom} g$ and g is measurable on E_2.
Then there exists an element E of S such that $E = \operatorname{dom}(f+g)$ and $f+g$ is measurable on E.

(54) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose $\operatorname{dom} f = A$. Then f is measurable on B if and only if f is measurable on $A \cap B$.

(55) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $\operatorname{dom} f = A$. Let c be a real number and B be an element of S. If f is measurable on B, then $c\,f$ is measurable on B.

4. Sequence of Extended Real Numbers

A sequence of extended reals is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to finite number if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists a real number g such that for every real number p if $0 < p$, then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \overline{\mathbb{R}}(g)| < p$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $+\infty$ if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let g be a real number. Suppose $0 < g$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $g \leq s_1(m)$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $-\infty$ if and only if the condition (Def. 10) is satisfied.

(Def. 10) Let g be a real number. Suppose $g < 0$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $s_1(m) \leq g$.

We now state two propositions:

(56) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $+\infty$. Then s_1 is not convergent to $-\infty$ and s_1 is not convergent to finite number.

(57) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $-\infty$. Then s_1 is not convergent to $+\infty$ and s_1 is not convergent to finite number.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent if and only if:

(Def. 11) s_1 is convergent to finite number, or convergent to $+\infty$, or convergent to $-\infty$.

Let s_1 be a sequence of extended reals. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields an extended real number and is defined by the conditions (Def. 12).

(Def. 12)(i) There exists a real number g such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$ and s_1 is convergent to finite number, or

(ii) $\lim s_1 = +\infty$ and s_1 is convergent to $+\infty$, or

(iii) $\lim s_1 = -\infty$ and s_1 is convergent to $-\infty$.

We now state a number of propositions:

(58) Let s_1 be a sequence of extended reals and r be a real number. Suppose that for every natural number n holds $s_1(n) = r$. Then s_1 is convergent to finite number and $\lim s_1 = r$.

(59) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$. If for every natural number n such that $n \in \operatorname{dom} F$ holds $0 \leq F(n)$, then $0 \leq \sum F$.

(60) Let L be a sequence of extended reals. Suppose that for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$. Then L is convergent and $\lim L = \sup \operatorname{rng} L$.

(61) For all sequences L, G of extended reals such that for every natural number n holds $L(n) \leq G(n)$ holds $\sup \operatorname{rng} L \leq \sup \operatorname{rng} G$.

(62) For every sequence L of extended reals and for every natural number n holds $L(n) \leq \sup \operatorname{rng} L$.

(63) Let L be a sequence of extended reals and K be an extended real number. If for every natural number n holds $L(n) \leq K$, then $\sup \operatorname{rng} L \leq K$.

(64) Let L be a sequence of extended reals and K be an extended real number. If $K \neq +\infty$ and for every natural number n holds $L(n) \leq K$, then $\sup \operatorname{rng} L < +\infty$.

(65) Let L be a sequence of extended reals. Suppose L is without $-\infty$. Then $\sup \operatorname{rng} L \neq +\infty$ if and only if there exists a real number K such that $0 < K$ and for every natural number n holds $L(n) \leq K$.

(66) Let L be a sequence of extended reals and c be an extended real number. Suppose that for every natural number n holds $L(n) = c$. Then L is convergent and $\lim L = c$ and $\lim L = \sup \operatorname{rng} L$.

(67) Let J, K, L be sequences of extended reals. Suppose that

(i) for all natural numbers n, m such that $n \leq m$ holds $J(n) \leq J(m)$,

(ii) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,

(iii) J is without $-\infty$,

(iv) K is without $-\infty$, and

(v) for every natural number n holds $J(n) + K(n) = L(n)$.

Then L is convergent and $\lim L = \sup \operatorname{rng} L$ and $\lim L = \lim J + \lim K$ and $\sup \operatorname{rng} L = \sup \operatorname{rng} K + \sup \operatorname{rng} J$.

(68) Let L, K be sequences of extended reals and c be a real number. Suppose $0 \leq c$ and L is without $-\infty$ and for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$. Then $\sup \operatorname{rng} K = \overline{\mathbb{R}}(c) \cdot \sup \operatorname{rng} L$ and K is without $-\infty$.

(69) Let L, K be sequences of extended reals and c be a real number. Suppose that

(i) $0 \leq c$,

(ii) for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$,

(iii) for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$, and

(iv) L is without $-\infty$.

Then

(v) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,

(vi) K is without $-\infty$ and convergent,

(vii) $\lim K = \sup \operatorname{rng} K$, and

(viii) $\lim K = \overline{\mathbb{R}}(c) \cdot \lim L$.

5. Sequence of Extended Real Valued Functions

Let X be a non empty set, let H be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and let x be an element of X. The functor $H\#x$ yields a sequence of extended reals and is defined as follows:

(Def. 13) For every natural number n holds $(H\#x)(n) = H(n)(x)$.

Let D_1, D_2 be sets, let F be a function from $\mathbb{N}$ into $D_1 \dot{\rightarrow} D_2$, and let n be a natural number. Then $F(n)$ is a partial function from D_1 to D_2.

Next we state the proposition

(70) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then there exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ such that

(i) for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = \operatorname{dom} f$,

(ii) for every natural number n holds $F(n)$ is non-negative,

(iii) for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leq F(m)(x)$, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$.

6. Integral of Non Negative Simple Valued Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int' f \, \mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 14) $$\int' f \, \mathrm{d}M = \begin{cases} \int\limits_X f \, \mathrm{d}M, & \text{if } \operatorname{dom} f \neq \emptyset, \\ 0_{\overline{\mathbb{R}}}, & \text{otherwise.} \end{cases}$$

The following propositions are true:

(71) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S and f is non-negative and g is non-negative. Then $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int' f + g \, \mathrm{d}M = \int' f {\restriction} \operatorname{dom}(f+g) \, \mathrm{d}M + \int' g {\restriction} \operatorname{dom}(f+g) \, \mathrm{d}M$.

(72) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is simple function in S and f is non-negative and $0 \leq c$. Then $\int' c \, f \, \mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int' f \, \mathrm{d}M$.

(73) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is simple function in S and f is non-negative and A misses B. Then $\int' f {\restriction} (A \cup B) \, \mathrm{d}M = \int' f {\restriction} A \, \mathrm{d}M + \int' f {\restriction} B \, \mathrm{d}M$.

(74) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $0 \leq \int' f \, \mathrm{d}M$.

(75) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) f is non-negative,
(iii) g is simple function in S,
(iv) g is non-negative, and
(v) for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$.

Then $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int' f \restriction \operatorname{dom}(f - g)\,\mathrm{d}M = \int' f - g\,\mathrm{d}M + \int' g \restriction \operatorname{dom}(f - g)\,\mathrm{d}M$.

(76) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
(i) f is simple function in S,
(ii) g is simple function in S,
(iii) f is non-negative,
(iv) g is non-negative, and
(v) for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$.

Then $\int' g \restriction \operatorname{dom}(f - g)\,\mathrm{d}M \leq \int' f \restriction \operatorname{dom}(f - g)\,\mathrm{d}M$.

(77) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose $0 \leq c$ and f is simple function in S and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = c$. Then $\int' f\,\mathrm{d}M = c \cdot M(\operatorname{dom} f)$.

(78) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int' f \restriction \text{EQ-dom}(f, \overline{\mathbb{R}}(0))\,\mathrm{d}M = 0$.

(79) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, B be an element of S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $M(B) = 0$ and f is non-negative. Then $\int' f \restriction B\,\mathrm{d}M = 0$.

(80) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, g be a partial function from X to $\overline{\mathbb{R}}$, F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and L be a sequence of extended reals. Suppose that g is simple function in S and for every set x such that $x \in \operatorname{dom} g$ holds $0 < g(x)$ and for every natural number n holds $F(n)$ is simple function in S and for every natural number n holds $\operatorname{dom} F(n) = \operatorname{dom} g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leq \lim(F\#x)$ and for every natural number n holds $L(n) = \int' F(n)\,\mathrm{d}M$. Then L is convergent and $\int' g\,\mathrm{d}M \leq \lim L$.

(81) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, g be a partial function from X to $\overline{\mathbb{R}}$, and F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. Suppose that g is simple function in S and g is non-negative and for every natural number n holds $F(n)$ is simple

function in S and for every natural number n holds $\operatorname{dom} F(n) = \operatorname{dom} g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leq \lim(F\#x)$. Then there exists a sequence G of extended reals such that for every natural number n holds $G(n) = \int' F(n)\,\mathrm{d}M$ and G is convergent and $\sup \operatorname{rng} G = \lim G$ and $\int' g\,\mathrm{d}M \leq \lim G$.

(82) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, F, G be sequences of partial functions from X into $\overline{\mathbb{R}}$, and K, L be sequences of extended reals. Suppose that for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = A$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $F(n)(x) \leq F(m)(x)$ and for every natural number n holds $G(n)$ is simple function in S and $\operatorname{dom} G(n) = A$ and for every natural number n holds $G(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $G(n)(x) \leq G(m)(x)$ and for every element x of X such that $x \in A$ holds $F\#x$ is convergent and $G\#x$ is convergent and $\lim(F\#x) = \lim(G\#x)$ and for every natural number n holds $K(n) = \int' F(n)\,\mathrm{d}M$ and $L(n) = \int' G(n)\,\mathrm{d}M$. Then K is convergent and L is convergent and $\lim K = \lim L$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. The functor $\int^{+} f\,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 15).

(Def. 15) There exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ and there exists a sequence K of extended reals such that
for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = \operatorname{dom} f$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$ and for every natural number n holds $K(n) = \int' F(n)\,\mathrm{d}M$ and K is convergent and $\int^{+} f\,\mathrm{d}M = \lim K$.

The following propositions are true:

(83) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $\int^{+} f\,\mathrm{d}M = \int' f\,\mathrm{d}M$.

(84) Let X be a non empty set, S be a σ-field of subsets of X, M be a

σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) there exists an element B of S such that $B = \operatorname{dom} g$ and g is measurable on B,

(iii) f is non-negative, and

(iv) g is non-negative.

Then there exists an element C of S such that $C = \operatorname{dom}(f + g)$ and $\int^{+} f + g \, \mathrm{d}M = \int^{+} f \restriction C \, \mathrm{d}M + \int^{+} g \restriction C \, \mathrm{d}M$.

(85) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $0 \leq \int^{+} f \, \mathrm{d}M$.

(86) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative. Then $0 \leq \int^{+} f \restriction A \, \mathrm{d}M$.

(87) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and A misses B. Then $\int^{+} f \restriction (A \cup B) \, \mathrm{d}M = \int^{+} f \restriction A \, \mathrm{d}M + \int^{+} f \restriction B \, \mathrm{d}M$.

(88) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $M(A) = 0$. Then $\int^{+} f \restriction A \, \mathrm{d}M = 0$.

(89) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int^{+} f \restriction A \, \mathrm{d}M \leq \int^{+} f \restriction B \, \mathrm{d}M$.

(90) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S. Suppose f is non-negative and $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $\int^{+} f \restriction (E \setminus A) \, \mathrm{d}M = \int^{+} f \, \mathrm{d}M$.

(91) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element E of S such that $E = \operatorname{dom} f$ and $E = \operatorname{dom} g$ and f is measurable on E and g is measurable on E,

(ii) f is non-negative,

(iii) g is non-negative, and

(iv) for every element x of X such that $x \in \operatorname{dom} g$ holds $g(x) \leq f(x)$.

Then $\int^{+} g \, \mathrm{d}M \leq \int^{+} f \, \mathrm{d}M$.

(92) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose $0 \leq c$ and there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $\int^{+} c\, f \, \mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int^{+} f \, \mathrm{d}M$.

(93) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A, and

(ii) for every element x of X such that $x \in \operatorname{dom} f$ holds $0 = f(x)$.

Then $\int^{+} f \, \mathrm{d}M = 0$.

7. Integral of Measurable Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int f \, \mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 16) $\int f \, \mathrm{d}M = \int^{+} \max_{+}(f) \, \mathrm{d}M - \int^{+} \max_{-}(f) \, \mathrm{d}M$.

We now state several propositions:

(94) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $\int f \, \mathrm{d}M = \int^{+} f \, \mathrm{d}M$.

(95) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int f \, \mathrm{d}M = \int^{+} f \, \mathrm{d}M$ and $\int f \, \mathrm{d}M = \int' f \, \mathrm{d}M$.

(96) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $0 \leq \int f \, \mathrm{d}M$.

(97) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and A misses B. Then $\int f{\restriction}(A \cup B) \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

(98) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative. Then $0 \leq \int f \restriction A \, \mathrm{d}M$.

(99) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f \restriction A \, \mathrm{d}M \leq \int f \restriction B \, \mathrm{d}M$.

(100) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $\int f \restriction A \, \mathrm{d}M = 0$.

(101) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S. If $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f \restriction (E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is integrable on M if and only if:

(Def. 17) There exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and $\int^{+} \max_{+}(f) \, \mathrm{d}M < +\infty$ and $\int^{+} \max_{-}(f) \, \mathrm{d}M < +\infty$.

One can prove the following propositions:

(102) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then $0 \leq \int^{+} \max_{+}(f) \, \mathrm{d}M$ and $0 \leq \int^{+} \max_{-}(f) \, \mathrm{d}M$ and $-\infty < \int f \, \mathrm{d}M$ and $\int f \, \mathrm{d}M < +\infty$.

(103) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose f is integrable on M. Then $\int^{+} \max_{+}(f \restriction A) \, \mathrm{d}M \leq \int^{+} \max_{+}(f) \, \mathrm{d}M$ and $\int^{+} \max_{-}(f \restriction A) \, \mathrm{d}M \leq \int^{+} \max_{-}(f) \, \mathrm{d}M$ and $f \restriction A$ is integrable on M.

(104) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is integrable on M and A misses B. Then $\int f \restriction (A \cup B) \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M + \int f \restriction B \, \mathrm{d}M$.

(105) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is integrable on M and $B = \operatorname{dom} f \setminus A$. Then $f \restriction A$ is integrable on M and $\int f \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M + \int f \restriction B \, \mathrm{d}M$.

(106) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(107) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is integrable on M, then $|\int f \, \mathrm{d}M| \leq \int |f| \, \mathrm{d}M$.

(108) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) $\operatorname{dom} f = \operatorname{dom} g$,

(iii) g is integrable on M, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| \leq g(x)$.

Then f is integrable on M and $\int |f| \, \mathrm{d}M \leq \int g \, \mathrm{d}M$.

(109) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\operatorname{dom} f \in S$ and $0 \leq r$ and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then $\int\limits_X f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(110) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\operatorname{dom} f \in S$ and $0 \leq r$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then $\int' f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(111) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then $f^{-1}(\{+\infty\}) \in S$ and $f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\})) = 0$ and $M(f^{-1}(\{-\infty\})) = 0$ and $f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\})) = 0$.

(112) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M.

(113) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(114) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M. Then $f + g$ is integrable on M.

(115) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is

integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f+g \, \mathrm{d}M = \int f \restriction E \, \mathrm{d}M + \int g \restriction E \, \mathrm{d}M$.

(116) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is integrable on M. Then $c\,f$ is integrable on M and $\int c\,f \, \mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\overline{\mathbb{R}}$, and let B be an element of S. The functor $\int\limits_B f \, \mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 18) $\int\limits_B f \, \mathrm{d}M = \int f \restriction B \, \mathrm{d}M$.

The following propositions are true:

(117) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be an element of S. Suppose f is integrable on M and g is integrable on M and $B \subseteq \operatorname{dom}(f+g)$. Then $f+g$ is integrable on M and $\int\limits_B f+g \, \mathrm{d}M = \int\limits_B f \, \mathrm{d}M + \int\limits_B g \, \mathrm{d}M$.

(118) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, c be a real number, and B be an element of S. Suppose f is integrable on M and f is measurable on B. Then $f \restriction B$ is integrable on M and $\int\limits_B c\,f \, \mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int\limits_B f \, \mathrm{d}M$.

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