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Some Properties of Some Special Matrices. Part II

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Summary. This article provides definitions of idempotent, nilpotent, involutory, self-reversible, similar, and congruent matrices, the trace of a matrix and their main properties.

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The terminology and notation used here are introduced in the following articles: [7], [3], [1], [9], [8], [6], [4], [2], [5], [11], and [10].

We adopt the following convention: n is a natural number, K is a field, and M_1, M_2, M_3, M_4, M_5, M_6 are matrices over K of dimension n.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is idempotent if and only if:

(Def. 1) $M_1 \cdot M_1 = M_1$.

We say that M_1 is 2-nilpotent if and only if:

(Def. 2) $$M_1 \cdot M_1 = \left(\begin{array}{ccc} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{array}\right)_K^{n \times n}.$$

We say that M_1 is involutory if and only if:

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(Def. 3) $M_1 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

We say that M_1 is self invertible if and only if:

(Def. 4) M_1 is invertible and $M_1^{\smile} = M_1$.

We now state a number of propositions:

(1) $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is idempotent and involutory.

(2) If $n > 0$, then $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times n}$ is idempotent and 2-nilpotent.

(3) If $n > 0$ and $M_2 = M_1^{\mathrm{T}}$, then M_1 is idempotent iff M_2 is idempotent.

(4) If M_1 is involutory, then M_1 is invertible.

(5) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2, then $M_1 \cdot M_1$ is permutable with $M_2 \cdot M_2$.

(6) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times n}$, then $M_1 + M_2$ is idempotent.

(7) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$, then $M_1 + M_2$ is idempotent.

(8) If M_1 is idempotent and M_2 is invertible, then $M_2^{\smile} \cdot M_1 \cdot M_2$ is idempotent.

(9) If $n > 0$ and M_1 is invertible and idempotent, then $M_1^{\smile}$ is idempotent.

(10) If M_1 is invertible and idempotent, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(11) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2, then $M_1 \cdot M_2$ is idempotent.

(12) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_3 = M_1^{\mathrm{T}} \cdot M_2^{\mathrm{T}}$, then M_3 is idempotent.

(13) If M_1 is idempotent and M_2 is idempotent and M_1 is invertible, then $M_1 \cdot M_2$ is idempotent.

(14) If $n > 0$ and M_1 is idempotent and orthogonal, then M_1 is symmetrical.

(15) If M_1 is idempotent and M_2 is idempotent and $M_2 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$, then $M_1 \cdot M_2$ is idempotent.

(16) If M_1 is idempotent and orthogonal, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(17) If $n > 0$ and M_1 is symmetrical and $M_2 = {M_1}^{\mathrm{T}}$, then $M_1 \cdot M_2$ is symmetrical.

(18) If $n > 0$ and M_1 is symmetrical and $M_2 = {M_1}^{\mathrm{T}}$, then $M_2 \cdot M_1$ is symmetrical.

(19) If M_1 is invertible and $M_1 \cdot M_2 = M_1 \cdot M_3$, then $M_2 = M_3$.

(20) If M_1 is invertible and $M_2 \cdot M_1 = M_3 \cdot M_1$, then $M_2 = M_3$.

(21) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$, then $M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$.

(22) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$, then $M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$.

(23) If M_1 is 2-nilpotent and permutable with M_2 and $n > 0$, then $M_1 \cdot M_2$ is 2-nilpotent.

(24) If $n > 0$ and M_1 is 2-nilpotent and M_2 is 2-nilpotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$, then $M_1 + M_2$ is 2-nilpotent.

(25) If M_1 is 2-nilpotent and M_2 is 2-nilpotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$ and $n > 0$, then $M_1 + M_2$ is 2-nilpotent.

(26) If M_1 is 2-nilpotent and $M_2 = {M_1}^{\mathrm{T}}$ and $n > 0$, then M_2 is 2-nilpotent.

(27) If M_1 is 2-nilpotent and idempotent, then $M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$.

(28) If $n > 0$, then $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n} \neq \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(29) If $n > 0$ and M_1 is 2-nilpotent, then M_1 is not invertible.

(30) If M_1 is self invertible, then M_1 is involutory.

(31) $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is self invertible.

(32) If M_1 is self invertible and idempotent, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(33) If M_1 is self invertible and symmetrical, then M_1 is orthogonal.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is similar to M_2 if and only if:

(Def. 5) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^{\smile} \cdot M_2 \cdot M$.

Let us notice that the predicate M_1 is similar to M_2 is reflexive and symmetric.

The following propositions are true:

(34) If M_1 is similar to M_2 and M_2 is similar to M_3 and $n > 0$, then M_1 is similar to M_3.

(35) If M_1 is similar to M_2 and M_2 is idempotent, then M_1 is idempotent.

(36) If M_1 is similar to M_2 and M_2 is 2-nilpotent and $n > 0$, then M_1 is 2-nilpotent.

(37) If M_1 is similar to M_2 and M_2 is involutory, then M_1 is involutory.

(38) If M_1 is similar to M_2 and $n > 0$, then $M_1 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is similar to $M_2 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

(39) If M_1 is similar to M_2 and $n > 0$, then $M_1 + M_1$ is similar to $M_2 + M_2$.

(40) If M_1 is similar to M_2 and $n > 0$, then $M_1 + M_1 + M_1$ is similar to $M_2 + M_2 + M_2$.

(41) If M_1 is invertible, then $M_2 \cdot M_1$ is similar to $M_1 \cdot M_2$.

(42) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then M_1 is invertible.

(43) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then ${M_1}^{\smile}$ is similar to ${M_2}^{\smile}$.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is congruent to M_2 if and only if:

(Def. 6) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^{\mathrm{T}} \cdot M_2 \cdot M$.

Next we state several propositions:

(44) If $n > 0$, then M_1 is congruent to M_1.

(45) If M_1 is congruent to M_2 and $n > 0$, then M_2 is congruent to M_1.

(46) If M_1 is congruent to M_2 and M_2 is congruent to M_3 and $n > 0$, then M_1 is congruent to M_3.

(47) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1$ is congruent to $M_2 + M_2$.

(48) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1 + M_1$ is congruent to $M_2 + M_2 + M_2$.

(49) If M_1 is orthogonal, then $M_2 \cdot M_1$ is congruent to $M_1 \cdot M_2$.

(50) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$, then M_1 is invertible.

(51) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$ and $M_5 = {M_1}^{\mathrm{T}}$ and $M_6 = {M_2}^{\mathrm{T}}$, then M_5 is congruent to M_6.

(52) If M_4 is orthogonal and $M_1 = {M_4}^{\mathrm{T}} \cdot M_2 \cdot M_4$, then M_1 is similar to M_2.

Let n be a natural number, let K be a field, and let M be a matrix over K of dimension n. The functor $\mathrm{Trace}(M)$ yields an element of K and is defined by:

(Def. 7) $\mathrm{Trace}(M) = \sum(\text{the diagonal of } M)$.

The following propositions are true:

(53) If $M_2 = {M_1}^{\mathrm{T}}$, then $\mathrm{Trace}(M_1) = \mathrm{Trace}(M_2)$.

(54) $\mathrm{Trace}(M_1 + M_2) = \mathrm{Trace}(M_1) + \mathrm{Trace}(M_2)$.

(55) $\mathrm{Trace}(M_1 + M_2 + M_3) = \mathrm{Trace}(M_1) + \mathrm{Trace}(M_2) + \mathrm{Trace}(M_3)$.

(56) $\mathrm{Trace}(\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{K}^{n \times n}) = 0_K$.

(57) If $n > 0$, then $\mathrm{Trace}(-M_1) = -\mathrm{Trace}(M_1)$.

(58) If $n > 0$, then $-\mathrm{Trace}(-M_1) = \mathrm{Trace}(M_1)$.

(59) If $n > 0$, then $\mathrm{Trace}(M_1 + -M_1) = 0_K$.

(60) If $n > 0$, then $\mathrm{Trace}(M_1 - M_2) = \mathrm{Trace}(M_1) - \mathrm{Trace}(M_2)$.

(61) If $n > 0$, then $\mathrm{Trace}((M_1 - M_2) + M_3) = (\mathrm{Trace}(M_1) - \mathrm{Trace}(M_2)) + \mathrm{Trace}(M_3)$.

(62) If $n > 0$, then $\mathrm{Trace}((M_1 + M_2) - M_3) = (\mathrm{Trace}(M_1) + \mathrm{Trace}(M_2)) - \mathrm{Trace}(M_3)$.

(63) If $n > 0$, then $\mathrm{Trace}(M_1 - M_2 - M_3) = \mathrm{Trace}(M_1) - \mathrm{Trace}(M_2) - \mathrm{Trace}(M_3)$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[4] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[5] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.
[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[8] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[9] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[10] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.
[11] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.

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