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A Theory of Matrices of Real Elements

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Summary. Here, the concept of matrix of real elements is introduced. This is defined as a special case of the general concept of matrix of a field. For such a real matrix, the notions of addition, subtraction, scalar product are defined. For any real finite sequences, two transformations to matrices are introduced. One of the matrices is of width 1, and the other is of length 1. By such transformations, two products of a matrix and a finite sequence are defined. Also the linearity of such product is shown.

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The papers [16], [19], [6], [3], [10], [18], [15], [1], [14], [12], [20], [7], [2], [17], [13], [22], [8], [11], [5], [4], [21], and [9] provide the terminology and notation for this paper.

1. Preliminaries

In this paper i, j are natural numbers.

We now state a number of propositions:

(1) For all real numbers r_1, r_2 and for all elements f_1, f_2 of $\mathbb{R}_{\rm F}$ such that $r_1 = f_1$ and $r_2 = f_2$ holds $r_1 + r_2 = f_1 + f_2$.

(2) For all real numbers r_1, r_2 and for all elements f_1, f_2 of $\mathbb{R}_{\rm F}$ such that $r_1 = f_1$ and $r_2 = f_2$ holds $r_1 \cdot r_2 = f_1 \cdot f_2$.

(3) For every finite sequence F of elements of $\mathbb{R}$ holds $F + -F = \langle \underbrace{0, \ldots, 0}_{\mathrm{len}\, F} \rangle$ and $F - F = \langle \underbrace{0, \ldots, 0}_{\mathrm{len}\, F} \rangle$.

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(4) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $F_1 - F_2 = F_1 + -F_2$.

(5) For every finite sequence F of elements of $\mathbb{R}$ holds $F - \langle \underbrace{0, \ldots, 0}_{\operatorname{len} F} \rangle = F$.

(6) For every finite sequence F of elements of $\mathbb{R}$ holds $\langle \underbrace{0, \ldots, 0}_{\operatorname{len} F} \rangle - F = -F$.

(7) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $F_1 - -F_2 = F_1 + F_2$.

(8) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $-(F_1 - F_2) = F_2 - F_1$.

(9) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $-(F_1 - F_2) = -F_1 + F_2$.

(10) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ and $F_1 - F_2 = \langle \underbrace{0, \ldots, 0}_{\operatorname{len} F_1} \rangle$ holds $F_1 = F_2$.

(11) For all finite sequences F_1, F_2, F_3 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ and $\operatorname{len} F_2 = \operatorname{len} F_3$ holds $F_1 - F_2 - F_3 = F_1 - (F_2 + F_3)$.

(12) For all finite sequences F_1, F_2, F_3 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ and $\operatorname{len} F_2 = \operatorname{len} F_3$ holds $F_1 + (F_2 - F_3) = (F_1 + F_2) - F_3$.

(13) For all finite sequences F_1, F_2, F_3 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ and $\operatorname{len} F_2 = \operatorname{len} F_3$ holds $F_1 - (F_2 - F_3) = (F_1 - F_2) + F_3$.

(14) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $F_1 = (F_1 + F_2) - F_2$.

(15) For all finite sequences F_1, F_2 of elements of $\mathbb{R}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $F_1 = (F_1 - F_2) + F_2$.

2. Matrices of Real Elements

The following propositions are true:

(16) Let K be a non empty groupoid, p be a finite sequence of elements of K, and a be an element of K. Then $\operatorname{len}(a \cdot p) = \operatorname{len} p$.

(17) Let r be a real number, f_3 be an element of $\mathbb{R}_{\mathrm{F}}$, p be a finite sequence of elements of $\mathbb{R}$, and f_4 be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. If $r = f_3$ and $p = f_4$, then $r \cdot p = f_3 \cdot f_4$.

(18) Let K be a field, a be an element of K, and A be a matrix over K. Then the indices of $a \cdot A =$ the indices of A.

(19) Let K be a field, a be an element of K, and M be a matrix over K. If $1 \leq i$ and $i \leq \operatorname{width} M$, then $(a \cdot M)_{\square,i} = a \cdot M_{\square,i}$.

(20) Let K be a field, a be an element of K, M be a matrix over K, and i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} M$, then $\operatorname{Line}(a \cdot M, i) = a \cdot \operatorname{Line}(M, i)$.

(21) Let K be a field and A, B be matrices over K. Suppose $\operatorname{width} A = \operatorname{len} B$. Then there exists a matrix C over K such that $\operatorname{len} C = \operatorname{len} A$ and $\operatorname{width} C = \operatorname{width} B$ and for all i, j such that $\langle i, j \rangle \in$ the indices of C holds $C_{i,j} = \operatorname{Line}(A, i) \cdot B_{\Box,j}$.

(22) Let K be a field, a be an element of K, and A, B be matrices over K. If $\operatorname{width} A = \operatorname{len} B$ and $\operatorname{len} A > 0$ and $\operatorname{len} B > 0$, then $A \cdot (a \cdot B) = a \cdot (A \cdot B)$.

Let A be a matrix over $\mathbb{R}$. The functor $(\mathbb{R} \to \mathbb{R}_{\rm F})A$ yielding a matrix over $\mathbb{R}_{\rm F}$ is defined as follows:

(Def. 1) $(\mathbb{R} \to \mathbb{R}_{\rm F})A = A$.

Let A be a matrix over $\mathbb{R}_{\rm F}$. The functor $(\mathbb{R}_{\rm F} \to \mathbb{R})A$ yielding a matrix over $\mathbb{R}$ is defined by:

(Def. 2) $(\mathbb{R}_{\rm F} \to \mathbb{R})A = A$.

We now state two propositions:

(23) Let D_1, D_2 be sets, A be a matrix over D_1, and B be a matrix over D_2. Suppose $A = B$. Let given i, j. If $\langle i, j \rangle \in$ the indices of A, then $A_{i,j} = B_{i,j}$.

(24) For every field K and for all matrices A, B over K holds the indices of $A + B =$ the indices of A.

Let A, B be matrices over $\mathbb{R}$. The functor $A + B$ yields a matrix over $\mathbb{R}$ and is defined by:

(Def. 3) $A + B = (\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})A + (\mathbb{R} \to \mathbb{R}_{\rm F})B)$.

One can prove the following two propositions:

(25) Let A, B be matrices over $\mathbb{R}$. Then $\operatorname{len}(A + B) = \operatorname{len} A$ and $\operatorname{width}(A + B) = \operatorname{width} A$ and for all i, j such that $\langle i, j \rangle \in$ the indices of A holds $(A + B)_{i,j} = A_{i,j} + B_{i,j}$.

(26) Let A, B, C be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} C = \operatorname{len} A$ and $\operatorname{width} C = \operatorname{width} A$ and for all i, j such that $\langle i, j \rangle \in$ the indices of A holds $C_{i,j} = A_{i,j} + B_{i,j}$. Then $C = A + B$.

Let A be a matrix over $\mathbb{R}$. The functor $-A$ yields a matrix over $\mathbb{R}$ and is defined as follows:

(Def. 4) $-A = (\mathbb{R}_{\rm F} \to \mathbb{R})(-(\mathbb{R} \to \mathbb{R}_{\rm F})A)$.

Let A, B be matrices over $\mathbb{R}$. The functor $A - B$ yielding a matrix over $\mathbb{R}$ is defined as follows:

(Def. 5) $A - B = (\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})A - (\mathbb{R} \to \mathbb{R}_{\rm F})B)$.

The functor $A \cdot B$ yielding a matrix over $\mathbb{R}$ is defined by:

(Def. 6) $A \cdot B = (\mathbb{R}_{\rm F} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\rm F})A \cdot (\mathbb{R} \to \mathbb{R}_{\rm F})B)$.

Let a be a real number and let A be a matrix over $\mathbb{R}$. The functor $a \cdot A$ yields a matrix over $\mathbb{R}$ and is defined as follows:

(Def. 7) For every element e_1 of $\mathbb{R}_{\rm F}$ such that $e_1 = a$ holds $a \cdot A = (\mathbb{R}_{\rm F} \to \mathbb{R})(e_1 \cdot (\mathbb{R} \to \mathbb{R}_{\rm F})A)$.

The following propositions are true:

(27) For every real number a and for every matrix A over $\mathbb{R}$ holds $\mathrm{len}(a \cdot A) = \mathrm{len}\, A$ and $\mathrm{width}(a \cdot A) = \mathrm{width}\, A$.

(28) For every real number a and for every matrix A over $\mathbb{R}$ holds the indices of $a \cdot A =$ the indices of A.

(29) Let a be a real number, A be a matrix over $\mathbb{R}$, and i_2, j_2 be natural numbers. If $\langle i_2, j_2 \rangle \in$ the indices of A, then $(a \cdot A)_{i_2,j_2} = a \cdot A_{i_2,j_2}$.

(30) For every real number a and for every matrix A over $\mathbb{R}$ such that $\mathrm{len}\, A > 0$ and $\mathrm{width}\, A > 0$ holds $(a \cdot A)^{\rm T} = a \cdot A^{\rm T}$.

(31) Let a be a real number, i be a natural number, and A be a matrix over $\mathbb{R}$. Suppose $\mathrm{len}\, A > 0$ and $i \in \mathrm{dom}\, A$. Then

(i) there exists a finite sequence p of elements of $\mathbb{R}$ such that $p = A(i)$, and

(ii) for every finite sequence q of elements of $\mathbb{R}$ such that $q = A(i)$ holds $(a \cdot A)(i) = a \cdot q$.

(32) For every matrix A over $\mathbb{R}$ holds $1 \cdot A = A$.

(33) For every matrix A over $\mathbb{R}$ holds $A + A = 2 \cdot A$.

(34) For every matrix A over $\mathbb{R}$ holds $A + A + A = 3 \cdot A$.

Let n, m be natural numbers. The functor $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{R}}$ yields a matrix over $\mathbb{R}$ and is defined by:

(Def. 8) $$\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{R}} = (\mathbb{R}_{\rm F} \to \mathbb{R})(\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{R}_{\rm F}}).$$

One can prove the following propositions:

(35) For all matrices A, B over $\mathbb{R}$ such that $\mathrm{len}\, B > 0$ holds $A - -B = A + B$.

(36) Let n, m be natural numbers and A be a matrix over $\mathbb{R}$. If $\mathrm{len}\, A = n$ and $\mathrm{width}\, A = m$ and $n > 0$, then $A + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{R}} = A$ and $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{R}} + A = A$.

(37) For all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ and $A = A + B$ holds $B = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{\operatorname{len} A \times \operatorname{width} A}$.

(38) For all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ and $A + B = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{\operatorname{len} A \times \operatorname{width} A}$ holds $B = -A$.

(39) For all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ and $B - A = B$ holds $A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{\operatorname{len} A \times \operatorname{width} A}$.

(40) For every real number a and for all matrices A, B over $\mathbb{R}$ such that $\operatorname{width} A = \operatorname{len} B$ and $\operatorname{len} A > 0$ and $\operatorname{len} B > 0$ holds $A \cdot (a \cdot B) = a \cdot (A \cdot B)$.

(41) Let a be a real number and A, B be matrices over $\mathbb{R}$. If $\operatorname{width} A = \operatorname{len} B$ and $\operatorname{len} A > 0$ and $\operatorname{len} B > 0$ and $\operatorname{width} B > 0$, then $(a \cdot A) \cdot B = a \cdot (A \cdot B)$.

(42) For every matrix M over $\mathbb{R}$ such that $\operatorname{len} M > 0$ holds $M + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{\operatorname{len} M \times \operatorname{width} M} = M$.

(43) For every real number a and for all matrices A, B over $\mathbb{R}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{len} A > 0$ holds $a \cdot (A + B) = a \cdot A + a \cdot B$.

(44) For every matrix A over $\mathbb{R}$ such that $\operatorname{len} A > 0$ holds $0 \cdot A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{\operatorname{len} A \times \operatorname{width} A}$.

Let x be a finite sequence of elements of $\mathbb{R}$. Let us assume that $\operatorname{len} x > 0$. The functor $\operatorname{ColVec2Mx} x$ yields a matrix over $\mathbb{R}$ and is defined as follows:

(Def. 9) $\operatorname{len} \operatorname{ColVec2Mx} x = \operatorname{len} x$ and $\operatorname{width} \operatorname{ColVec2Mx} x = 1$ and for every j such that $j \in \operatorname{dom} x$ holds $(\operatorname{ColVec2Mx} x)(j) = \langle x(j) \rangle$.

The following three propositions are true:

(45) Let x be a finite sequence of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. If $\operatorname{len} x > 0$, then $M = \operatorname{ColVec2Mx} x$ iff $M_{\square,1} = x$ and $\operatorname{width} M = 1$.

(46) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 =$

$\operatorname{len} x_2$ and $\operatorname{len} x_1 > 0$ holds $\mathrm{ColVec2Mx}(x_1 + x_2) = \mathrm{ColVec2Mx}\, x_1 + \mathrm{ColVec2Mx}\, x_2$.

(47) For every real number a and for every finite sequence x of elements of $\mathbb{R}$ such that $\operatorname{len} x > 0$ holds $\mathrm{ColVec2Mx}(a \cdot x) = a \cdot \mathrm{ColVec2Mx}\, x$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{LineVec2Mx}\, x$ yielding a matrix over $\mathbb{R}$ is defined as follows:

(Def. 10) $\operatorname{width} \mathrm{LineVec2Mx}\, x = \operatorname{len} x$ and $\operatorname{len} \mathrm{LineVec2Mx}\, x = 1$ and for every j such that $j \in \operatorname{dom} x$ holds $(\mathrm{LineVec2Mx}\, x)_{1,j} = x(j)$.

The following propositions are true:

(48) Let x be a finite sequence of elements of $\mathbb{R}$ and M be a matrix over $\mathbb{R}$. Then $M = \mathrm{LineVec2Mx}\, x$ if and only if the following conditions are satisfied:

(i) $\mathrm{Line}(M, 1) = x$, and

(ii) $\operatorname{len} M = 1$.

(49) For every finite sequence x of elements of $\mathbb{R}$ such that $\operatorname{len} x > 0$ holds $(\mathrm{LineVec2Mx}\, x)^{\mathrm{T}} = \mathrm{ColVec2Mx}\, x$ and $(\mathrm{ColVec2Mx}\, x)^{\mathrm{T}} = \mathrm{LineVec2Mx}\, x$.

(50) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_1 > 0$ holds $\mathrm{LineVec2Mx}(x_1 + x_2) = \mathrm{LineVec2Mx}\, x_1 + \mathrm{LineVec2Mx}\, x_2$.

(51) For every real number a and for every finite sequence x of elements of $\mathbb{R}$ holds $\mathrm{LineVec2Mx}(a \cdot x) = a \cdot \mathrm{LineVec2Mx}\, x$.

Let M be a matrix over $\mathbb{R}$ and let x be a finite sequence of elements of $\mathbb{R}$. The functor $M \cdot x$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 11) $M \cdot x = (M \cdot \mathrm{ColVec2Mx}\, x)_{\square,1}$.

The functor $x \cdot M$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 12) $x \cdot M = \mathrm{Line}(\mathrm{LineVec2Mx}\, x \cdot M, 1)$.

Next we state a number of propositions:

(52) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} A > 0$ and if $\operatorname{width} A > 0$ and if $\operatorname{len} A = \operatorname{len} x$ or $\operatorname{width}(A^{\mathrm{T}}) = \operatorname{len} x$, then $A^{\mathrm{T}} \cdot x = x \cdot A$.

(53) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\operatorname{len} A > 0$ and if $\operatorname{width} A > 0$ and if $\operatorname{width} A = \operatorname{len} x$ or $\operatorname{len}(A^{\mathrm{T}}) = \operatorname{len} x$, then $A \cdot x = x \cdot A^{\mathrm{T}}$.

(54) Let A, B be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. Let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{width} A$, then $(A + B)_{\square,i} = A_{\square,i} + B_{\square,i}$.

(55) Let A, B be matrices over $\mathbb{R}$. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. Let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} A$, then $\mathrm{Line}(A+$

$B, i) = \mathrm{Line}(A, i) + \mathrm{Line}(B, i)$.

(56) Let a be a real number, M be a matrix over $\mathbb{R}$, and i be a natural number. If $1 \leq i$ and $i \leq \mathrm{width}\, M$, then $(a \cdot M)_{\Box,i} = a \cdot M_{\Box,i}$.

(57) Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ and $\mathrm{width}\, A = \mathrm{len}\, x_1$ and $\mathrm{len}\, x_1 > 0$ and $\mathrm{len}\, A > 0$, then $A \cdot (x_1 + x_2) = A \cdot x_1 + A \cdot x_2$.

(58) Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ and $\mathrm{len}\, A = \mathrm{len}\, x_1$ and $\mathrm{len}\, x_1 > 0$, then $(x_1 + x_2) \cdot A = x_1 \cdot A + x_2 \cdot A$.

(59) Let a be a real number, x be a finite sequence of elements of $\mathbb{R}$, and A be a matrix over $\mathbb{R}$. If $\mathrm{width}\, A = \mathrm{len}\, x$ and $\mathrm{len}\, x > 0$ and $\mathrm{len}\, A > 0$, then $A \cdot (a \cdot x) = a \cdot (A \cdot x)$.

(60) Let a be a real number, x be a finite sequence of elements of $\mathbb{R}$, and A be a matrix over $\mathbb{R}$. If $\mathrm{len}\, A = \mathrm{len}\, x$ and $\mathrm{len}\, x > 0$ and $\mathrm{width}\, A > 0$, then $(a \cdot x) \cdot A = a \cdot (x \cdot A)$.

(61) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\mathrm{width}\, A = \mathrm{len}\, x$ and $\mathrm{len}\, x > 0$ and $\mathrm{len}\, A > 0$, then $\mathrm{len}(A \cdot x) = \mathrm{len}\, A$.

(62) Let x be a finite sequence of elements of $\mathbb{R}$ and A be a matrix over $\mathbb{R}$. If $\mathrm{len}\, A = \mathrm{len}\, x$ and $\mathrm{len}\, x > 0$ and $\mathrm{width}\, A > 0$, then $\mathrm{len}(x \cdot A) = \mathrm{width}\, A$.

(63) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\mathrm{len}\, A = \mathrm{len}\, B$ and $\mathrm{width}\, A = \mathrm{width}\, B$ and $\mathrm{width}\, A = \mathrm{len}\, x$ and $\mathrm{len}\, A > 0$ and $\mathrm{len}\, x > 0$, then $(A + B) \cdot x = A \cdot x + B \cdot x$.

(64) Let x be a finite sequence of elements of $\mathbb{R}$ and A, B be matrices over $\mathbb{R}$. If $\mathrm{len}\, A = \mathrm{len}\, B$ and $\mathrm{width}\, A = \mathrm{width}\, B$ and $\mathrm{len}\, A = \mathrm{len}\, x$ and $\mathrm{width}\, A > 0$ and $\mathrm{len}\, x > 0$, then $x \cdot (A + B) = x \cdot A + x \cdot B$.

(65) Let n, m be natural numbers and x be a finite sequence of elements of $\mathbb{R}$. If $\mathrm{len}\, x = m$ and $n > 0$ and $m > 0$, then $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m} \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(66) Let n, m be natural numbers and x be a finite sequence of elements of $\mathbb{R}$. If $\mathrm{len}\, x = n$ and $n > 0$ and $m > 0$, then $x \cdot \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{R}}^{n \times m} = \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

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