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NON-STANDARD METHOD OF DISCRETIZATION ON THE EXAMPLE OF HAAVELMO GROWTH CYCLE MODEL

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Abstract

In the theory of economics most models describing economic growth make use of differential equations. The examples are Solow's and Haavelmo's models. However, when they are used by econometricians many questions arise. Firstly, economic data are presented in discrete form, which implies the use of difference equations. Secondly, the mode transition from continuous form to the discrete one in order to estimate its parameters is still controversial. It has been observed for some time that standard (classical) discretization methods of differential equations often produce difference equations that do not share their dynamics (for example produce chaotic behavior).

The essence of above-mentioned problems and proposal of solving them will be presented on the basis of Haavelmo model.

Keywords: Computer simulation, Haavelmo Growth Cycle model, Kahan's discretization method.

JEL classification: C02, C15, C62, Y4.

Introduction

Systems of ordinary differential equations (ODEs) appear in various applications. Few ODEs can be solved explicitly. We often have to visualize the solutions by using discretization methods and computer programming. After applying a discretization (or numerical) method to system of ODEs, it results in a system of difference equations, or discrete time system.

It has been observed for some time that the standard (classical) discretization methods of differential equations often produce difference equations that do not share their dynamics. An illustrative example is the logistic difference equation:

$$\frac{dx}{dt} = x(1-x) \qquad\qquad x(0) = x_0 .$$

Euler's discretization scheme produces the logistic difference equation:

$$x_{n+1} = x_n + hx_n(1-x_n), \qquad x(0) = x_0 , \qquad n = 0,1,2... ,$$

which possesses a remarkably different dynamics such as period-doubling bifurcation route to chaos[1]. In difference equation, a steady state (fixed point) solutions are defined by the value that satisfies the relation $x_{n+1} = x_n$. In ODEs, a steady state solution satisfies $dx/dt = 0$. Fixed points of the differential equations are kept by the newly formed difference equation. Therefore, both equations have the same fixed points, 0 and 1. This simple example shows how use of the Euler method changes dynamic behavior of the model.

Because in the economics the problem of discretization of continuous models is frequent, particularly in economic data, it is crucial to look for a discretization method that preserves dynamic properties of the model. Studies show that the most fruitful discretization methods are those of Mickens[2] and Kahan[3]. The article is aimed at application of one of the above-mentioned methods (Kahan's method) to Haavelmo model.

1. Haavelmo Model

In 1954 Haavelmo[4] proposed the growth cycle model:

$$\frac{\dot{L}}{L} = \alpha - \beta\frac{L}{Y},\ \alpha,\beta > 0 , \tag{1}$$

$$Y = KL^a,\ K > 0,\ 0 < a < 1 , \tag{2}$$

where L and Y are functions of time ad the dot denotes a time derivative.

Equation (2) has real output Y produced with a constant output elasticity a under decreasing returns by the labour force L. In equation (1) N can be seen to grow autonomously at the proportional rate α, minus rate that depends inversely on per capita output. Therefore, the growth increases with per capita output and is bounded above by α.

By joining equations (1) and (2) we obtain the following differential equation:

$$\dot{L} = \alpha L - \frac{\beta}{K} L^{2-a}, \tag{3}$$

which describes dynamics of expenses of labour force.

It is easy to show that it has two stationary solutions (using Bernoulli substitution):

non-stable $L_1^* = 0$ and asymptotically stable $L_2^* = \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$.

Thanks to the knowledge of stable, stationary solution we can write general solution described by the following time path:

$$L(t) = \left(\frac{\left(K(L(0))^{a-1} - \frac{\beta}{\alpha} \right) e^{\alpha(a-1)t} + \frac{\beta}{\alpha}}{K} \right)^{\frac{1}{a-1}}. \tag{4}$$

The dynamics are quite simple.

If the initial condition $L(0) > \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$, then both L and Y will decrease monotonously ($L(0) < \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$, then both L and Y will increase monotonously) approaching their unique steady state values $\left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$ and $K\left(\frac{\alpha K}{\beta}\right)^{\frac{a}{1-a}}$ respectively.

2. Discretization

2.1. Haavelmo -Stutzer Model

In 1980 Stutzer[5] considered non-linear model of growth cycle presented earlier for constant time by Haavelmo[6].

In his work, Stutzer replaced differential operator in Haavelmo model by finite difference (he replaced $\dot{L}$ by $L_{t+1} - L_t$), (Euler discretization with $h = 1$) obtaining non-linear difference equation of the first order describing behaviour of employment in time:

$$L_{t+1} = (1+\alpha)L_t - \frac{\beta}{K} L_t^{\,2-a} \tag{6}$$

Performing dynamic analysis for the equation (6) will allow to obtain the optimal employment level. However, it is inconvenient for the above-mentioned form, therefore by means of transformation:

$$L_t = \left(\frac{K(1+\alpha)}{\beta}\right)^{\frac{1}{1-a}} x_t\,, \tag{7}$$

we can write equation (6) as:

$$x_{t+1} = (1+\alpha)x_t\left(1 - x_t^{1-a}\right), \tag{8}$$

where $x_t = L_t\left(\frac{K(1+\alpha)}{\beta}\right)^{-\frac{1}{1-a}}$.

The equation (8) is still a non-linear difference equation of the first order and its form enables further dynamic analysis.

In order to find equilibrium state and intervals, for which there is convergence to the state of equilibrium, we will use properties of local bifurcations. We will use α as a decision parameter.

Let

$$\Phi(x) = (1+\alpha)x\left(1 - x^{1-a}\right), \tag{9}$$

we solve equation: $\Phi(x) = x$, or $(1+\alpha)x\left(1 - x^{1-a}\right) = x$. As a result, we obtain two equilibrium points:

$$c_0^* = 0\,,\ \ c_1^* = \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{1-a}},$$

or, after transformation with the use of (7):

$$L_1^* = 0\,,\ \ L_2^* = \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}.$$

In further considerations we omit the trivial solution. Next, we analyse the convergence of the equation (8) to the equilibrium point L_2^*.

According to the definition of local bifurcations, the following condition must be satisfied: $|\Phi'(c_1)| < 1$, from which it results that L_2^* is a stable equilibrium point, if $\alpha \in \left(0, \frac{2}{1-a}\right]$, while for $\alpha \in \left(0, \frac{1}{1-a}\right]$ it is asymptotically stable equilibrium point and for $\alpha \in \left(\frac{1}{1-a}, \frac{2}{1-a}\right]$, the convergence to c_1^* is oscillatory.

Let the parameter $\alpha = 0.2$, $\beta / K = 1$ while an initial value $L_0 = 0.6$.

Along with the above-mentioned considerations, the stable equilibrium point of equation (7) is $L^* = (\alpha)^{\frac{10}{8}}$, for the parameter $\alpha \in (0, 2.5]$. For the $\alpha \in (0, 1.25]$ convergence to the equilibrium level is monotonous and for the $\alpha \in (1.25, 2.5]$ – oscillatory.

The necessary condition for reaching the equilibrium level was $\alpha \leq 2.5$ at a fixed level of the remaining parameters. We can ask the question: what happens if the parameter α exceeds this value? The obtained results show that if the parameter rises on, the value of function moves between two levels cyclically. The more the parameter increases, the faster the duplications follow. The model's behaviour becomes chaotic. Therefore it can be showed that for the values of parameters accepted earlier, the model behaves chaotically in the sense of Li-York theorem[7] for the decision parameter $\alpha \in (2.5; \approx 3.37)$.

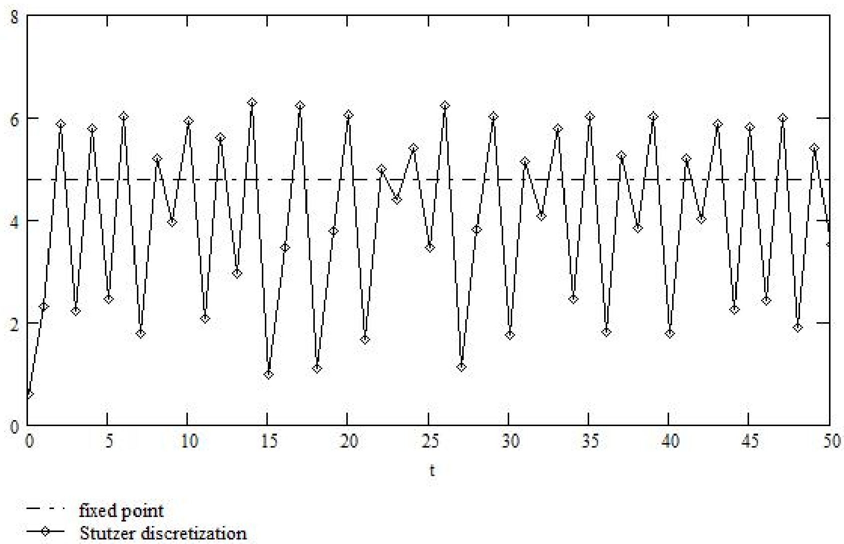

Fig.1. Haavelmo-Stutzer Model chaotic behaviour for the parameter α = 3.5
Source: own study.

2.2. The Euler's Method

There are numerous discretization schemes in the Numerical Analysis literature. The simplest numerical scheme is the forward Euler in which $\frac{dx}{dt}$ is replaced by $\frac{x_{t+h}-x_t}{h}$, where h is the step-size of numerical method. Making this replacement in equation (3), and letting $t = nh$, $L_t = L_{nh} = L_n$, yields the difference system:

$$L_{t+1} = L_t + \alpha \cdot h \cdot L_t - \frac{\beta}{K} \cdot h \cdot L_t^{2-a}. \tag{10}$$

Using Theorem 1, of Roeger[8] we can establish the dynamic consistency of equation (10). In this paper also the following Theorem 1 was proven.

Theorem 1. Let p_0 be a fixed point of the system of differential equations (3). Then after we apply the Euler's method, the stability of p_0 will become

(i) unstable, if p_0 was unstable,

(ii) stable, if p_0 was stable and all of the eigenvalues of Jacobian matrix $Df(p_0)$ are located inside the disk $|z+1/h|<1/h$,

(iii) unstable, if p_0 was stable and some of eigenvalues of the Jacobian matrix $Df(p_0)$ are located outside the disk $|z+1/h|<1/h$.

Moreover, if p_0 is stable, we can always preserve the stability of p_0 by choosing the step-size h small enough such that all of eigenvalues of Jacobian matrix $Df(p_0)$ are located inside the disc $|z+1/h|<1/h$.

By Theorem 1 we know that in order to preserve the local stability of the fixed point $p_0 = L_2^* = \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$, we need to have $f'(p_0) = -\alpha + \alpha \cdot a$ located inside the disc $|z+1/h|<1/h$. That is, $|-\alpha + \alpha \cdot a + 1/h| < 1/h$. Therefore, we have $h < \frac{2}{\alpha(1-a)}$. Then $h > \frac{2}{\alpha(1-a)}$ we expect the point $p_0 = L_2^* = \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$ to be unstable.

If we choose the step-size h to be $0 < h < \frac{2}{\alpha(1-a)}$, (for $a = 0.2$, $L_0 = 0.6$ and $\alpha = 3.5$ $0 < h < 0.714$) the solution should approach the stable fixed point $p_0 = L_2^* = \left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$.

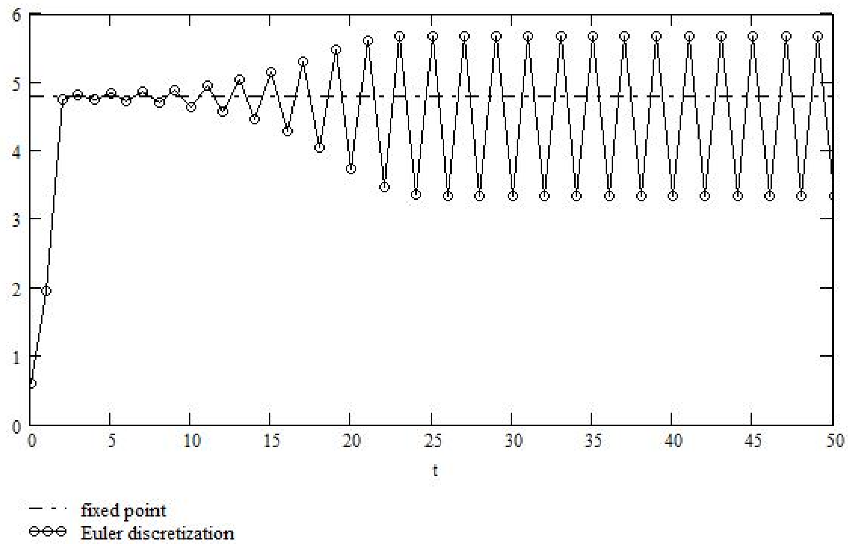

Fig.2. Applying Euler's method to the Haavelmo model (3), we can see that when we choose $h = 0.8 > 0.714$, the fixed points L_2^* become unstable. The solution oscillates between two different values

Source: own study.

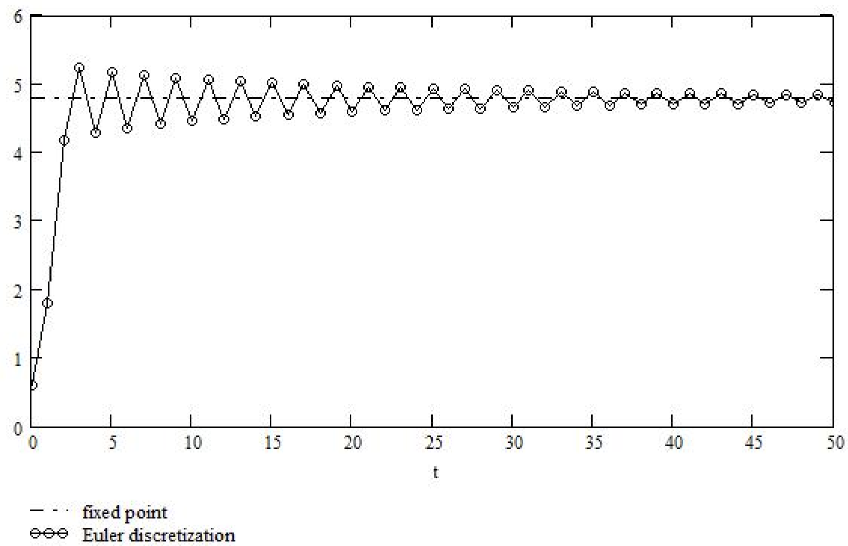

Fig.3. When applying Euler's method to Haavelmo model (3), we can see that when we choose $h = 0.7 < 0.714$, the fixed point L_2^* is stable

Source: own study.

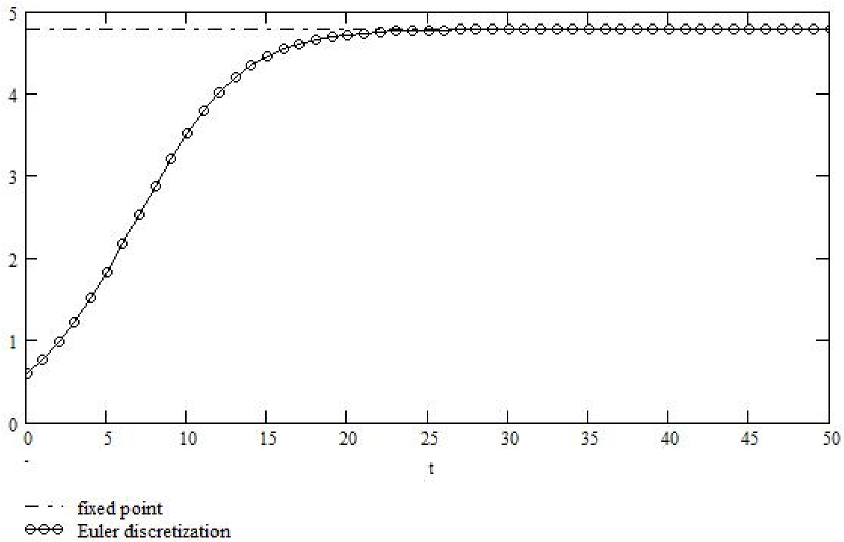

Fig.4. When applying Euler's method to Haavelmo model (3), we can see that when we choose $h = 0.1$, (small step-size), the fixed point L_2^* is stable, and the coherence takes place fast

Source: own study.

2.3. The Kahan's Method

As the alternative to the Euler method, this article proposes Kahan's method[9]. If system of autonomous differential equations can be written as:

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x}) \,, \tag{11}$$

then we can write Kahan's method as:

$$\boldsymbol{X} = \boldsymbol{x} + h \cdot \left(1 - \frac{h}{2} Df(\boldsymbol{x})\right)^{-1} \cdot f(\boldsymbol{x}), \tag{12}$$

where:

$\boldsymbol{x} = (x_1, x_2, \ldots x_n)^T$,

$Df(\boldsymbol{x})$ – the Jacobian matrix of $f(\boldsymbol{x})$ evaluated at $\boldsymbol{x}$,

$\boldsymbol{X} = \boldsymbol{x}_{t+1}$ – the new system of difference equations following the Kahan discretization for the variable $\boldsymbol{x_t} = ((x_1)_t, (x_2)_t, \ldots (x_n)_t)^T$.

By applying Kahan's method to Haavelmo model (3) we get the following numerical scheme:

$$L_{t+1}=L_t\cdot\frac{-2K(L_t)^a-\alpha K(L_t)^a h+\beta a L_t h}{-2K(L_t)^a+\alpha K(L_t)^a h-2\beta L_t h+\beta a L_t h} \tag{13}$$

Using Theorem 2 of Roeger[10] we can establish the dynamic consistency of the equation (13). In this paper also the following Theorem 2 was proven.

Theorem 2. Let p_0 be a fixed point of the differential equations (3) and h is small enough such that $\|Df(p_0)\|<2/h$. Then the fixed point p_0 of the differential equations (3) is locally asymptotically stable (unstable) if and only if the fixed point p_0 of the difference equations (13) is locally asymptotically stable (unstable).

In order to keep the fixed point $p_0=L_2^*=\left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$ to be stable by Theorem 2, we need to have $|f'(p_0)|<2/h$, ($|\alpha-2\alpha+a\cdot\alpha|<2/h$) i.e. $h<\frac{2}{\alpha(1-a)}$. If we choose the step-size h to be $0<h<\frac{2}{\alpha(1-a)}$ (for $a=0.2$, $L_0=0.6$ and $\alpha=3.5$ $0<h<0.714$), the solution should approach the stable fixed point $L_2^*=\left(\frac{\alpha K}{\beta}\right)^{\frac{1}{1-a}}$.

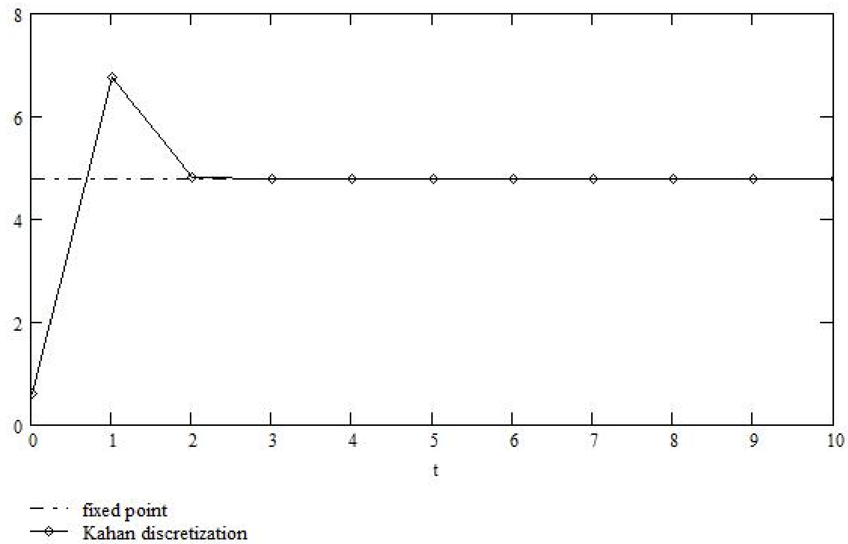

Fig. 5. When applying Kahan's method to the Haavelmo model (3), we can see that when we choose $h=0.7<0.714$, the fixed point L_2^* is stable. The solution approaches L_2^* much faster then the Euler's method

Source: own study.

As we can see in the Figure 4 Kahan discretization results, even for $h = 0.7 < 0.714$, in a very fast convergence to the equilibrium point. When *h* exceeds 0.714 the convergence rate is increasingly slower and, as a result, when $h = 0.9$ a complete lack of convergence takes place (a part of iteration takes complex values).

Conclusions

In this paper two discretization methods were used for the Haavelmo growth cycle model: Euler's (case: with *h*=1 – Stutzer Model, and with $h \neq 1$**)** and Kahan's ones. It was shown that even at small step *h*, the stable fixed point L_2^* for the initial differential equation is unstable for the difference equation created as a result of the Euler method. The second discretization method – Kahan's method preserves dynamic properties of the model.

Notes

[1] Mickens (1994).

[2] *Ibidem.*

[3] Kahan (1993).

[4] Haavelmo (1954).

[5] Stutzer (1980).

[6] Haavelmo (1954).

[7] Elaydi (2007), p.96.

[8] Roeger (2006).

[9] Kahan (1993).

[10] Roeger (2006).

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