Wydział Nauk Ekonomicznych i Zarządzania
Uniwersytetu Szczecińskiego

Folia Oeconomica Stetinensia
DOI: 10.2478/v10031-007-0003-6

ESTIMATION OF APPROXIMATE VALUES OF THE OPTIMUM POINTS ON EFFICIENT PORTFOLIOS CURVE

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Received 24 April 2007, Accepted 17 October 2007

Abstract

In the paper a method is found for estimating approximate optimum points on efficient portfolios curve (risk-profit) that are connected with exponential utility functions being very frequently preferred in practice by investors.

Keywords: utility functions, efficient portfolios, approximate optimum points.

JEL classification: C61.

Introduction

The selection of points optimal for investors on the curve of efficient portfolios is of close connection with utility functions preferred by them. The practice shows that they frequently adopt as their own the utility functions in the following form[1]:

$$U = R - a_1 e^{\sigma + b_1 \sigma^2} \tag{1}$$

where:

a_1, b_1 – parameters of utility function,

σ – standard deviation of return rate,

R – expected return rate.

The set of minimum risk within the co-ordinate system of standard deviation and return rate takes on a form of hyperbola of the following equation[2]:

$$\frac{(\sigma - m)^2}{a^2} - \frac{(R - n)^2}{b^2} = 1, \tag{2}$$

where:

a – semi-transverse axis of hyperbola,

b – semi-conjugate axis of hyperbola,

[m, n] – vector of hyperbola shift,

σ – standard deviation of return rate,

R – rate of return.

After small transformations, the equation (2) assumes the following form:

$$R = \frac{b}{a}\sqrt{(\sigma - m)^2 - a^2} + n. \tag{3}$$

This equation presents a set of efficient portfolios.

Let us see how to find common points of utility curves of the form (1) and of efficient portfolios curve (3) which is the upper part of hyperbola within the co-ordinate system of standard deviation and return rate[3].

With this end in view, let us assume that we examine the utility curve corresponding with utility $U = k$. The following equation should be solved:

$$k + a_1 e^{\sigma + b_1 \sigma^2} = \frac{b}{a}\sqrt{(\sigma - m)^2 - a^2} + n \tag{4}$$

or equivalently, after transformations:

$$a_1^{\,2}a^2e^{2(\sigma+b_1\sigma^2)}+2a^2a_1(k-n)e^{\sigma+b_1\sigma^2}-b^2\sigma^2+2b^2\sigma m-b^2(m^2-a^2)-a^2(2kn-k^2-n^2)=0$$

or in short:

$$\widetilde{a}e^{2(\sigma+b_1\sigma^2)}+\widetilde{b}e^{\sigma+b_1\sigma^2}+\widetilde{c}\sigma^2+\widetilde{d}\sigma+\widetilde{e}=0 \qquad (5)$$

where:

$$\widetilde{a}=a_1^{\,2}a^2,\widetilde{b}=2a^2a_1(k-n),\widetilde{c}=-b^2,\ \widetilde{d}=2b^2m,\ \widetilde{e}=-b^2(m^2-a^2)-a^2(2kn-k^2-n^2).$$

When k = 0, the equation (5) takes on the following form:

$$a_1^{\,2}a^2e^{2(\sigma+b_1\sigma^2)}-2a^2a_1ne^{\sigma+b_1\sigma^2}-b^2\sigma^2+2b^2\sigma m-b^2(m^2-a^2)+a^2n^2=0. \qquad (6)$$

As it can be already seen from the beginning, the equation (4) can not be solved accurately for all utility functions of the form (1). In order to estimate approximate roots of this equation, one may use the Bolzano-Cauchy theorem which says that if a given continuous function assumes different values at the extreme limits of a given closed interval, there is a point inside that interval at which a given function sets to zero[4]. Obviously, the transformation under our general examination is continuous.

1. Estimation of approximate values of the optimum points

When attempting to find at rough estimate the contact point between efficient portfolios curve and the curve corresponding with a given utility value, we can equate function derivatives. The equation (5) can be transformed to the following form:

$$\widetilde{a}e^{2(\sigma+b_1\sigma^2)}+\widetilde{b}e^{\sigma+b_1\sigma^2}=-\widetilde{c}\sigma^2-\widetilde{d}\sigma-\widetilde{e}\ . \qquad (7)$$

Let

$$f_1(\sigma)=\widetilde{a}e^{2(\sigma+b_1\sigma^2)}+\widetilde{b}e^{\sigma+b_1\sigma^2}, \qquad g_1(\sigma)=-\widetilde{c}\sigma^2-\widetilde{d}\sigma-\widetilde{e}\ .$$

The transformation $f_1(\sigma)$ is an exponential function, while the $g_1(\sigma)$ one is a quadratic function, the plot of which is a parabola with arms directed upwards (beacuse $-\widetilde{c}>0$).

When calculating derivatives of functions f_1 and g_1 with k = 0, we obtain:

$$f_1'(\sigma)=2a^2a_1^{\,2}(1+2b_1\sigma)e^{2(\sigma+b_1\sigma^2)}-2a^2a_1\,n(1+2b_1\sigma)e^{\sigma+b_1\sigma^2}, \quad g_1'(\sigma)=2b^2\sigma-2b^2m.$$

When comparing the derivatives after small transformations, we obtain the following equation:

$$f(\sigma) = (1+2b_1\sigma)e^{2(\sigma+b_1\sigma^2)} - \frac{n(2b_1+1)}{a_1}e^{\sigma+b_1\sigma^2} = (\frac{b}{aa_1})^2(\sigma - m) = g(\sigma)\,. \qquad (8)$$

When finding the value of standard deviation, at which the derivatives are equal, we will at the same time come across at the slopes of tangent lines (see Figure 1).

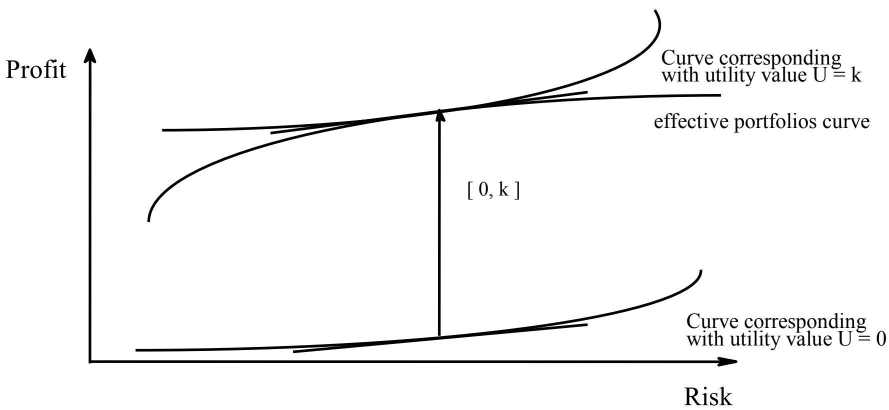

Fig. 1. Finding the points of contact of efficient portfolios curve and the curve corresponding with a given utility value

Source: own research.

In order to apply the Bolzano-Cauchy theorem, let us adopt the following notation:

$$h(\sigma) = f(\sigma) - g(\sigma) = (1+2b_1\sigma)e^{2(\sigma+b_1\sigma^2)} - \frac{n(2b_1+1)}{a_1}e^{\sigma+b_1\sigma^2} - (\frac{b}{aa_1})^2(\sigma - m)\,. \qquad (9)$$

The function h(σ) is continuous. As it can be easily noticed, the equation (9) can not be solved accurately either. One may also find the approximate value of contact point coordinates differently, using the Taylor series expression expansion[5]. Therefore, we obtain:

$$(1+2b_1\sigma)e^{2(\sigma+b_1\sigma^2)} - \frac{n(2b_1+1)}{a_1}e^{\sigma+b_1\sigma^2} - (\frac{b}{aa_1})^2\sigma + (\frac{b}{aa_1})^2 m = (1-2\frac{n}{a_1}-\frac{n}{a_1}) + (\frac{b}{aa_1})^2 +$$

$$[(2-2\frac{n}{a_1})b_1 - \frac{n}{a_1} + 2 - (\frac{b}{aa_1})^2]\sigma + [-2\frac{n}{a_1}b_1^2 + (-2\frac{n}{a_1}+6)b_1 + 2 - \frac{1}{2}\frac{n}{a_1}]\sigma^2 +$$

$$[(-2\frac{n}{a_1}+4)b_1^2 + (8-\frac{4}{3}\frac{n}{a_1})b_1 + \frac{4}{3} - \frac{1}{6}\frac{n}{a_1}]\sigma^3 + [-\frac{n}{a_1}b_1^3 + (-\frac{3}{2}\frac{n}{a_1}+10)b_1^2 + (\frac{20}{3}-\frac{7}{12}\frac{n}{a_1})b_1 -$$

$$\frac{1}{24}\frac{n}{a_1} + \frac{2}{3}]\sigma^4 = 0. \qquad (10)$$

Substituting the solution from the equation (10) to the equation (3), we obtain the approximate value of optimum point (σ, R). The value of utility function at which we have the point of contact is generally calculated from the following relationship (see Figure 1):

$$k = [\frac{b}{a}\sqrt{(\sigma - m)^2 - a^2} + n] - a_1 \cdot e^{\sigma + b_1\sigma^2} \tag{11}$$

where:

$a_1 \cdot e^{\sigma + b_1\sigma^2}$ (corresponds with $U = 0$).

2. Empirical example

Let us examine now an exemplary set of efficient portfolios consisting of the stocks of BRE, Elektrim and Universal companies (quoted on the Warsaw Stock Exchange, Poland). Basing on the two-year weekly data from January 1994 to January 1996, an efficient portfolios curve has been obtained (see Figure 2).

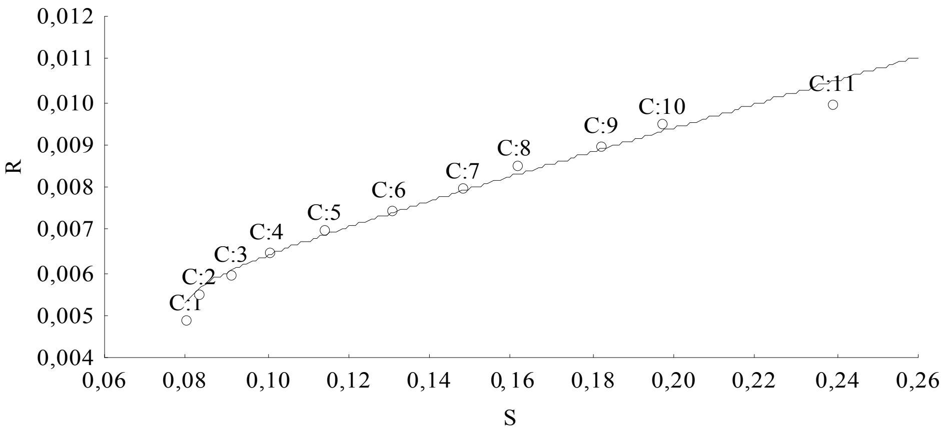

Fig. 2. Fragment of efficient portfolios curve for BRE, Elektrim and Universal companies obtained with the Hooke-Jeeves-Quasi-Newton method (method of arbitrary function of loss) with a fitting degree of 99.51%

Source: own research.

On Figure 2, we assumed $\sigma = S$.

From the received model we obtain: b= 0.0011, a = 0.041, m = 0.039, n = 0.0051.

$$R(\sigma) = \frac{0,0011}{0,041}\sqrt{(\sigma - 0,039)^2 - 0,041^2} + 0,0051.$$

Let an investor prefer a utility function: $U = R(\sigma) - 0,01014e^{\sigma + 0.01\sigma^2}$.

Then, with U= k = 0, we obtain:

$$f(\sigma) = (1 + 2 \cdot 0.01\sigma)e^{2(\sigma + 0.01\sigma^2)} - (\frac{2 \cdot 0.0051 \cdot 0.01}{0.01014} + \frac{0.0051}{0.01014})e^{\sigma + 0.01\sigma^2}$$

$$g(\sigma) = 7(\sigma - 0.039).$$

When illustrating the common part of curves $f(\sigma)$ and g(σ), we obtain (see Figure 3):

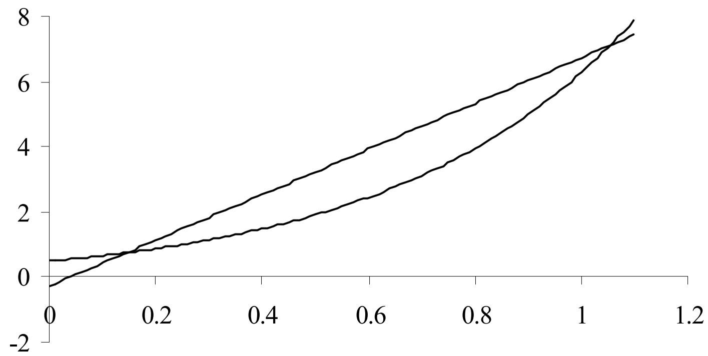

Fig. 3. Illustration of the common part of the plots of functions $f(\sigma)$ and g(σ)
Source: own research.

When using the Bolzano-Cauchy theorem for the function h(σ) = $f(\sigma)$ – g(σ) and allowing for Figure 3, we obtain:

$$h(0.14) = f(0.14) - g(0.14) = 0.03,$$

$$h(0.15) = f(0.15) - g(0.15) = -0.019$$

It can be seen from the above that standard deviation of the contact point of efficient portfolios curve and the examined utility curve is situated within the interval < 0.14, 0.15 >, because the function h(σ) changes its sign. When keeping on approximating, we obtain:

$$h(0.1461) = 3.098 \cdot 10^{-4}$$

$$h(0.1462) = -1.775 \cdot 10^{-4}.$$

Therefore, standard deviation of the contact point is to be found within the interval < 0.1461, 0.1462 >. It can be certainly approximated further. If we assume, however, that it amounts to 0.1462, this result differs from the real one by less than 0.0001.

Let us check now what a rate of return and a utility value corresponds with the standard deviation σ = 0.1462. After calculations, we obtain from (3):

$$R(0.1462) = \frac{0.001}{0.041}\sqrt{(0.1462 - 0.039)^2 - 0.041^2} + 0.0051 = 7.756 * 10^{-3},$$

and

$$0.01014 \cdot e^{0.1462 + 0.01 * 0.1462^2} = 0.012 \qquad \text{(with U = 0)},$$

which gives from (11):

$$k = 7.756 \cdot 10^{-3} - 0.012 = -0.004244.$$

Thus, the return rate in our case amounts to $R = 7.756 \cdot 10^{-3}$ with the standard deviation σ = 0.1462 and the utility value $U = k$ = -0.004244. In other words, our approximate point of contact $(0.1462, 7.756 \cdot 10^{-3})$ occurs when the utility U = -0.004244. At the same time, we can observe that the obtained utility value is at rough estimate the largest value at a given utility function.

When substituting the data from Figure 2: a = 0.041, b = 0.0011, m = 0.039, n = 0.0051, and utility function parameters: $a_1 = 0.01014, b_1 = 0.01$, to the equation (10), we obtain:

$$(1 - 2\frac{n}{a_1} - \frac{n}{a_1}) + (\frac{b}{aa_1})^2 = 0.76, \qquad (2 - 2\frac{n}{a_1})b_1 - \frac{n}{a_1} + 2 - (\frac{b}{aa_1})^2 = -5.494,$$

$$-2\frac{n}{a_1}b_1^2 + (-2\frac{n}{a_1} + 6)b_1 + 2 - \frac{1}{2}\frac{n}{a_1} = 1.798, \quad (-2\frac{n}{a_1} + 4)b_1^2 + (8 - \frac{4}{3}\frac{n}{a_1})b_1 + \frac{4}{3} - \frac{1}{6}\frac{n}{a_1} = 1.323,$$

$$-\frac{n}{a_1}b_1^3 + (-\frac{3}{2}\frac{n}{a_1} + 10)b_1^2 + (\frac{20}{3} - \frac{7}{12}\frac{n}{a_1})b_1 - \frac{1}{24}\frac{n}{a_1} + \frac{2}{3} = 0.71,$$

and the solution of equation:

$$0.71\sigma^4 + 1.323\sigma^3 + 1.798\sigma^2 - 5.494\sigma + 0.76 = 0$$

assumes the following form:

$$\sigma_1 = 1.1409, \ \sigma_2 = 0.1461, \ \sigma_3 = 1.5752 + 1.9847i, \ \sigma_4 = -1.5752 - 1.9847i.$$

The second solution, which practically agrees with the solution obtained by us when basing on the Bolzano-Cauchy theorem, deserves a particular attention. Certainly, the use of the Taylor's series with the components, the indices exponents of which are larger than or equal to 5, does not give an opportunity to obtain an accurate solution in each case, which results from the fact that there are no general formulas for calculating the roots of equations larger than or equal to 5.

Conclusions

Summing up, the method of searching for approximate optimum points on the risk-profit curve from the point of view of exponential utility functions preferred by investor presented above allows for finding approximate values of these points. In the investment practice, however, the presented approximation is fully sufficient.

Notes

1 See Tarczyński (1997).

2 See Haugen (1993).

3 See *Ibidem*.

4 See Fichtenholz (1969).

5 See Chiang (1984), Fichtenholz (1969).

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