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PROBABILITY OF EXERCISE OF OPTION

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Abstract

To estimate the risk the investors take when investing their money in stocks or stock options one must study if the option is exercised or not. From the point of view of a call option writer, especially those uncovered, one should study the probability of the exercise of option by a holder. The method presented in the paper enables to estimate risk connected with investment in options. In the assessment of risk that is born when investing money in stocks or options it is interesting whether the option will be exercised or not. From the writers' point of view, particularly those without coverage, it could be necessary to analyse probability of the exercise of options by buyers. The described method allows to assess at any time of call option duration whether the investor can be certain of the result of their investment. It can be applied also for the option strategies. In the paper the author has attempted to estimate the risk of call option and to estimate the probability of profit achievement in the case of long strangle option application. Investors using option strategies are able to do preliminary analysis of options and to minimize risk of their investment through choosing a proper date and price of exercise.

Keywords: risk, option strategies, mathematical methods, probability.

JEL classification: C02, C65, G11, G32.

Introduction

To estimate the risk the investors take when investing their money in stocks or stock options one must study if the option is exercised or not. From the point of view of a call option writer, especially those uncovered, one should study the probability of the exercise of option by a holder. The purpose of this work is to construct and analyse the function of probability of exercised option in reliance of classic concepts of calculus of probability, mathematical analysis and statistics. It is a proposal of a certain measure of risk of an investment in options. This measure permits to estimate the probability of an event that the call option will be exercised. Further, this method will be applied in the long strangle strategy option.

1. Function of the probability of the exercised of option

Let us assume that we buy a European call stock option. Let stock strike price on a purchase day be S, option premium S_0, exercise price – S_w. Let S_r be stock strike price at the expiration date of option, t_k – the expiration date, r_1 and r_2 – the annual rate of return calculated with the use of continuous accumulation of interest[1]. Let us assume that strike prices of an underlying instrument have a distribution of density $y = g(x)$[2]. Let us study the probability of an event that the option will be exercised, which means that: $S_r \geq S_w$.

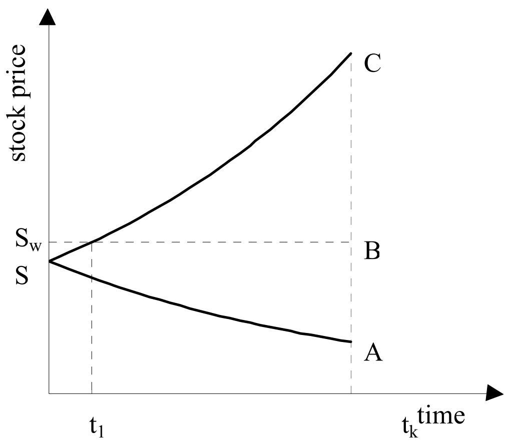

Fig. 1. The area of possible stock strike prices for $r_2 \leq 0 \leq r_1$ and $S \leq S_w < Se^{r_1 t_k}$
Source: own research.

Let us study the case where $r_2 \le 0 \le r_1$. For the above assumptions stock price for time $t \in \langle 0, t_k \rangle$ can have values within the area limited by the curves (Figure 1):

$$y = Se^{r_1 t} \tag{1}$$

and

$$y = Se^{r_2 t}, \tag{2}$$

where:

t – time in years since the option purchase,

r_1 – maximum annual rate of return,

r_2 – minimum annual rate of return.

Assuming that exercise date of an option is infinite meaning t_k increases without bound and depending on the position of curves (1) and (2) and the line $y = S_w$ we must consider two cases:

1. If $S \le S_w < Se^{r_1 t_k}$, then probability of event A that an option will be exercised in time t_k can be calculated as follows:

$$P(A) = \frac{CB}{CA}. \tag{3}$$

If t_k increases without bound the probability of an event that the option will be exercised can be shown as function $\hat{P}$ of variable t as follows:

$$\hat{P}(t) = \begin{cases} 0 & \text{for } t \in \langle 0, t_1 \rangle \\ \dfrac{\int_{S_w}^{Se^{r_1 t}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t \in (t_1, \infty) \end{cases}, \tag{4}$$

where:

$$t_1 = \frac{\ln \frac{S_w}{S}}{r_1}. \tag{5}$$

2. If $Se^{r_2 t_k} < S_w \le S$, then line $y = S_w$ crosses only curve $y = Se^{r_2 t_k}$. Function $\hat{P}$ looks as follows:

$$\hat{P}(t)=\begin{cases} 1 & \text{for } t\in\langle 0,t_1\rangle \\ \dfrac{\int_{S_w}^{Se^{r_1t}} g(x)dx}{\int_{Se^{r_2t}}^{Se^{r_1t}} g(x)dx} & \text{for } t\in(\mathrm{t}_1,\infty) \end{cases}, \tag{6}$$

where:

$$t_1=\frac{\ln\dfrac{S_w}{S}}{r_2}. \tag{7}$$

2. The function of probability of profit in option strategies

The above study can be conducted for any option strategy. In case of option strategy we do not use the term “strategy exercise” but we can talk about strategy giving profit. Hence function $\hat{P}$ will now be called profit function in strategy. We will not consider here option premiums paid for options that are part of a strategy. You should consider that if the purchase of call option gives profit then the purchase of put option would not give profit.

Let us study combination type strategy[3] in which there is one long call and one long put. Let exercise price of call option be S_{w2}, and exercise of put option be S_{w1}, the stock strike price on a purchase day of options be S, t_k - options expiry date, r_1 and r_2 - annual rate of return calculated with the use of continuous compounding. Let us also assume that the strategy is built on the basis of the same underlying instrument. Then interest rates r_1 and r_2 are the same for all options that are part of a strategy. Let us also assume that $r_1 > r_2$. Strategy gives profit if underlying instrument price is higher than S_{w2} or lower than S_{w1}.

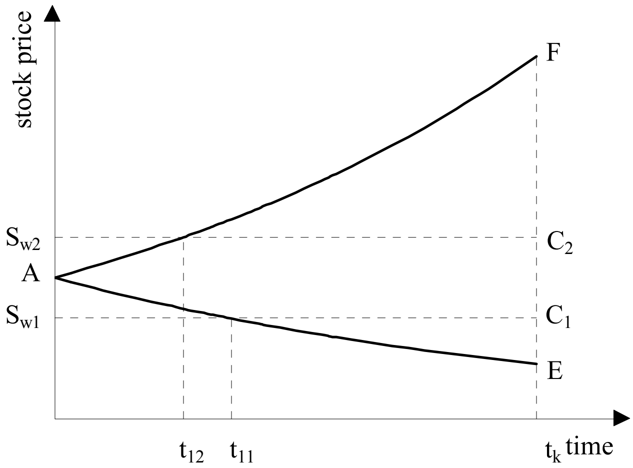

Fig. 2. The area of possible stock strike prices for long strangle strategy
Source: own research.

If $S_{w2} > S_{w1}$, then we have long strangle strategy[4] (Figure 2). We must study a few possibilities. To show the function of profit probability in a strategy we will consider one case only.

Let $S_{w1} < S < S_{w2}$, then:

$$t_{11} = \frac{\ln \frac{S_{w1}}{S}}{r_2}, \tag{8}$$

$$t_{12} = \frac{\ln \frac{S_{w2}}{S}}{r_1}. \tag{9}$$

Then for $t_{12} < t_{11}$ function $\hat{P}$ of probability of profit in a strategy can be represented as follows:

$$\hat{P}(t) = \begin{cases} 0 & \text{for } t \in \langle 0, t_{12}) \\ \dfrac{\int_{S_{w2}}^{Se^{r_1 t}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t \in \langle t_{12}, t_{11}) \\ 1 - \dfrac{\int_{S_{w1}}^{S_{w2}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t \in \langle t_{11}, \infty) \end{cases}, \tag{10}$$

where t_{11}, t_{12} are shown consecutively by (8) and (9).

3. The area of uncertainty of option exercise

The above considerations concerned the situation of time $t = 0$, on the option purchase day. Let us see what can happen in any time $t \in \langle 0, t_k \rangle$. Stock price can have the value from a definite area. There can be situations when we are certain that an option will be exercised, the situation when we are certain an option will not be exercised and uncertain situations, which are those when we cannot predict if an option is going to be exercised or not.

Let strike price on a purchase day of an option be *S,* maximum annual return – r_1, minimum annual rate of – r_2. Then the area of possible option prices is limited from the top with the curve (1) and from the bottom with the curve (2).

Let us assume that at time $t_2 \in \langle 0, t_k \rangle$ stock strike price is S_2 and it has such value that at time t_k it will reach value S_w, which can be represented as follows:

$$S_2 e^{r_1(t_k - t_2)} = S_w . \tag{11}$$

This situation is shown in Figure 3. Points (t_2, S_2) conforming to (11) are a curve below which there are points for which it is impossible to cross line $y = S_w$ upwards in time $t \le t_k$ measured since the moment $t = 0$. The equation for that curve considering (11) can be as follows:

$$y = S_w e^{r_1(t - t_k)} . \tag{12}$$

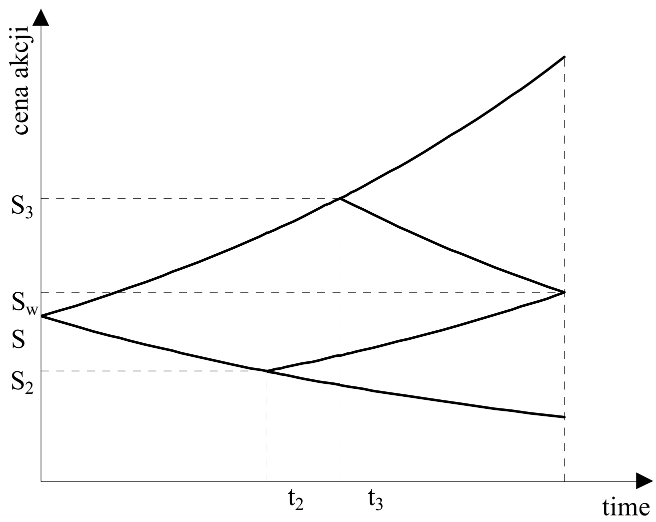

Fig. 3. The position of border curves for call stock option for $t_2 \le t_3$
Source: own research.

Similarly, we find the curve above which there are points for which it is impossible to cross line $y = S_w$ downwards in time $t \le t_k$ measured since the moment $t = 0$. The equation for that curve can be shown as follows:

$$y = S_w e^{r_2(t-t_k)}. \tag{13}$$

Curves (1), (2), (12) and (13) can be called **border curves.** Crossing points of border curves represented by equations (2) and (12) and equations (1) and (13) will be called **border points**. Abscissas of those points can be t_2 and t_3 respectively. The point of a smaller abscissa will be called **the first border point** and the point of a bigger abscissa – **the second border point**.

To find coordinates t_2 and t_3 we should use the fact that border points of those abscissas are the crossing points of border curves of equations (2) and (12) and equations (1) and (13). After calculating we get:

$$t_2 = \frac{\ln\frac{S}{S_w} + r_1 t_k}{r_1 - r_2}, \tag{14}$$

$$t_3 = \frac{\ln\frac{S_w}{S} - r_2 t_k}{r_1 - r_2}. \tag{15}$$

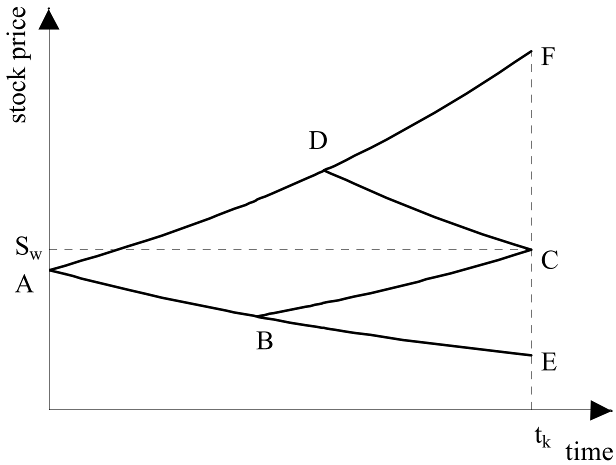

Fig. 4. Areas of uncertainty of an exercise and certainty of an exercise of European call stock option
Source: own research.

Figure 4 shows the following areas:

1. Area *ABCD*, which we will call the area of uncertainty. Let us assume that curves *BC* and *DC* belong to that area.
2. Area *DCF* – the area in which it is certain that an option will be exercised.
3. Area *BEC* – the area in which it is certain that an option will not be exercised.

Conditions for existence of t_2 and t_3

For t_2 and t_3 to exist the following conditions must be met:

$$0 \le t_2 \le t_k , \tag{16}$$

$$0 \le t_3 \le t_k . \tag{17}$$

Let us analyse the above inequalities. After putting formula (14) to formula (16) and formula (15) to formula (17) we arrive at:

$$0 \le \frac{\ln \frac{S_w}{S} - r_2 t_k}{r_1 - r_2} \le t_k , \tag{18}$$

$$0 \le \frac{\ln \frac{S}{S_w} + r_1 t_k}{r_1 - r_2} \le t_k . \tag{19}$$

After a few more modifications from formula (18) and (19) we arrive at:

$$Se^{r_2 t_k} \le S_w \le Se^{r_1 t_k} . \tag{20}$$

Formula (20) is the condition for the existence of t_2 and t_3.

Property 1

The sum of abscissas of border points is equal to time to maturity, which means:

$$t_2 + t_3 = t_k \,. \tag{21}$$

Proof:

Using formulas (14) and (15) we arrive at:

$$t_2 + t_3 = \frac{\ln \frac{S}{S_w} + r_1 t_k}{r_1 - r_2} + \frac{\ln \frac{S_w}{S} - r_2 t_k}{r_1 - r_2} = \frac{(r_1 - r_2)t_k}{r_1 - r_2} = t_k \,.$$

On the basis of given S_w, S, r_1, r_2, t_k one can define which area we are in at moment $t \in \langle 0, t_k \rangle$. It is apparent that with time the size of the area of uncertainty changes. It can be described with function P_1 [5], which defines the probability of being in the area of uncertainty at moment t.

1. If $t_2 \le t_3$, then function P_1 can be represented as follows:

$$P_1(t) = \begin{cases} 1 & \textit{for } \mathrm{t} \in \langle 0, t_2) \\ \dfrac{\int_{S_w e^{r_1 (t - t_k)}}^{S e^{r_1 t}} g(x)dx}{\int_{S e^{r_2 t}}^{S e^{r_1 t}} g(x)dx} & \textit{for } t \in \langle t_2, t_3) \\ \dfrac{\int_{S_w e^{r_1 (t - t_k)}}^{S_w e^{r_2 (t - t_k)}} g(x)dx}{\int_{S e^{r_2 t}}^{S e^{r_1 t}} g(x)dx} & \textit{for } t \in \langle t_3, t_k \rangle \end{cases} , \tag{22}$$

where t_2 and t_3 are defined by formula (14) and (15) respectively.

2. If $t_2 > t_3$, then function P_1 can be shown as:

$$P_1(t)=\begin{cases} 1 & \text{for } t\in\langle 0,t_3) \\ \dfrac{\int_{Se^{r_2 t}}^{S_w e^{r_2(t-t_k)}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t\in\langle t_3,t_2) \\ \dfrac{\int_{S_w e^{r_1(t-t_k)}}^{S_w e^{r_2(t-t_k)}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t\in\langle t_2,t_k\rangle \end{cases}, \tag{23}$$

where t_2 and t_3 are defined by formula (14) and (15) respectively.

4. The area of uncertainty for chosen option strategies

We will now define the areas of uncertainty of a strategy, which are the areas in which we are not certain if we will make profit or will suffer loss.

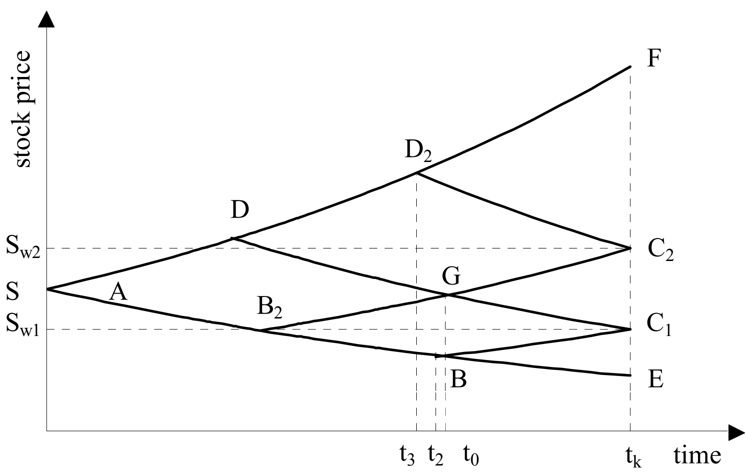

Fig. 5. Areas of uncertainty and certainty of option exercise in case long strangle strategy for $S_{w1} < S_{w2}$ and $t_3 \le t_2$
Source: own research.

Let us consider combination type strategy with the one long call option at exercise price S_{w2} and one long put option at exercise price S_{w1}. S, t_k, r_1, r_2 have been earlier defined in the paper.

If $S_{w1} < S_{w2}$, then we arrive at long strangle strategy. Area $AB_1C_1GC_2D_2$ (Figure 5) is the area of uncertainty about profit or loss in a given strategy. Being in area GC_1C_2 one can be certain that the strategy will be loss. The sum of areas B_1EC_1 and D_2C_2F is the area of certainty that long strangle strategy will being profit.

Time coordinate t_3 of point D_2 is condition the point of crossing of curves: (1) and

$$D_2C_2:\ y = S_{w2}e^{r_2(t-t_k)}. \tag{24}$$

Time coordinate t_2 of point B_1 is condition the point of crossing of curves: (2) and

$$B_1C_1:\ y = S_{w1}e^{r_1(t-t_k)}. \tag{25}$$

Time coordinate t_0 of point G is condition the point of crossing of curves:

$$B_2C_2:\ y = S_{w2}e^{r_1(t-t_k)}, \tag{26}$$

$$D_1C_1:\ y = S_{w1}e^{r_2(t-t_k)}. \tag{27}$$

We arrive at:

$$t_2 = \frac{r_1t_k - \ln\frac{S_{w1}}{S}}{r_1 - r_2}, \tag{28}$$

$$t_3 = \frac{\ln\frac{S_{w2}}{S} - r_2t_k}{r_1 - r_2}, \tag{29}$$

$$t_0 = t_k - \frac{\ln\frac{S_{w2}}{S_{w1}}}{r_1 - r_2}. \tag{30}$$

It is clear that with time the size of the area of uncertainty changes. It can be described with function P_I, which defines the probability of being in the area of uncertainty at moment t. Hence, there are 6 possible cases. Let us study one of them.

Let $t_3 \le t_2 \le t_0$. Function P_I can be represented as follows:

$$P_1(t)=\begin{cases} 1 & \text{for } t\in\langle 0,t_3) \\ \dfrac{\int_{Se^{r_2 t}}^{S_{w2}e^{r_2(t-t_k)}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t\in\langle t_3,t_2) \\ \dfrac{\int_{S_{w1}e^{r_1(t-t_k)}}^{S_{w2}e^{r_2(t-t_k)}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t\in\langle t_2,t_0) \\ \dfrac{\int_{S_{2s}e^{r_1(t-t_k)}}^{S_{w2}e^{r_2(t-t_k)}} g(x)dx+\int_{S_{w2}e^{r_1(t-t_k)}}^{S_{w1}e^{r_2(t-t_k)}} g(x)dx}{\int_{Se^{r_2 t}}^{Se^{r_1 t}} g(x)dx} & \text{for } t\in\langle t_0,t_k\rangle \end{cases}, \tag{31}$$

where t_2, t_3, t_0 are given by formulas (28), (29), (30) respectively.

Property 2

The sum of abscissas of border points t_0, t_2, t_3 is equal to doubled time to maturity:

$$t_0+t_2+t_3=2t_k\,. \tag{32}$$

Proof:

Applying formulas (28), (29), (30) we arrive at:

$$t_0+t_2+t_3=t_k-\frac{\ln\frac{S_{w2}}{S_{w1}}}{r_1-r_2}+\frac{r_1t_k-\ln\frac{S_{w1}}{S}}{r_1-r_2}+\frac{\ln\frac{S_{w2}}{S}-r_2t_k}{r_1-r_2}=\frac{t_k(r_1-r_2)+r_1t_k-r_2t_k}{r_1-r_2}=2t_k\,.$$

In practice formulas (28)-(30) can be applied to primary estimation of investment risk in time $t=0$. Example of calculation is in Table 1.

Table 1. Border points of selected option strategies

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
r_1	0,13	0,13	0,13	0,07
r_2	-0,16	-0,16	-0,16	-0,1
t_k (in months)	6	1	6	6
S	5	5	5	5
S_{w1}	4,9	4,9	4,6	4,6
S_{w2}	5,1	5,1	5,4	5,4

t_2 (in days)	106	39	184	251
t_3 (in days)	124	41	195	269
t_0 (in days)	130	-20	-19	-160

Source: own research.

In the first strategy the areas of certainty of profit will show up in 106 and 126 day, the area of certainty of loss – in 130 day. In the case of strategies 2-4 we have: $t_2 > t_k$, $t_3 > t_k$ and $t_0 < 0$. This implies loss for an investor (lack of areas: of certainty of profit and of uncertainty).

Conclusions

Created probability functions concerning one call option and combination strategy allow us to determine if an option will be exercised or not or if a strategy chosen by an investor will bring profit. The method presented in the paper is based on defining those changes on the basis of historical data. Those predictions can differ from real changes in future but in many situations this is quick method to apply especially if there is no well developed financial market. The calculated probability is for an investor risk measure, which he can accept or reject.

The other method concerns the possibility of determining at any time to maturity of call option (or option strategy) if an investor can be certain about the result of his investment. At first the investors are in the area of uncertainty. It may happen that values of border points will be beyond interval $\langle 0, t_k \rangle$. For investor this is information about probability of loss. With time the area of uncertainty decreases and there appear the areas of certainty. Writers of uncovered call options can in this way analyse the probability of option exercise by a holder. Investors applying option strategies can initially analyse options and through the choice of exercise date and exercise prices they can limit the risk of their investment.

Notes

[1] Ross (2003), pp.41-42.

[2] Probability of exercise options for log-normal distribution of prices in Stolorz (2005).

[3] More in this subject in Jajuga, Jajuga (1998), pp.199-207.

[4] More about long strangle strategy in Hull (2003), p.196.

[5] Plots of function P_1 for uniform distribution in: Stolorz (2006), Stolorz, Tarczyński (2002).

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