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OPERATIONS ON RISK VARIABLES

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Abstract

In the article the author considers and analyzes operations and functions on risk variables. She takes into account the following variables: the sum of risk variables, its product, multiplication by a constant, division, maximum, minimum and median of a sum of random variables. She receives the formulas for probability distribution and basic distribution parameters. She conducts the analysis for dependent and independent random variables. She proposes the examples of the situations in the economy and production management of risk modelled by these operations. The analysis is conducted with the way of mathematical proving. Some of the formulas presented are taken from the literature but others are the permanent results of the author.

Keywords: risk variables, operations, risk measurement, risk management

JEL classification: F39, C00

Introduction

Risk variable may be defined as a random variable because it achieves particular values of loss or win with the specified probability (Jajuga, 2009). In this article the author will define the fundamental operations on risk variables for dependent and independent cases. She will also present the formulas on probability of the received risk variables. These variables are equal the performed operations.

Risk is an important issue especially nowadays, after the well-known financial crisis, which began in USA and is named subprime crisis. It caused serious losses, especially in finance sector and was a reason of big recession in some other countries. According to economists, one of the main causes of the subprime crisis was wrong risk management (Buszkowska, 2015), which is the subject of this paper. There were also other well-known crises in the world in recent history. The crisis of the 1980's, the Great Depression and the Internet Bubble in the 20th century are the characteristic examples. Risk is an important issue not only in finance but it is also present in almost every area. We can mention the main areas of risk like catastrophes, banks, insurance (Jajuga, 2009), companies – operational risk (Staniec, 2008), trade – market risk (Jajuga, 2009), people – personal and human risk (Hanisz, 2010), nature and environment protection (Staniec, 2008), The results of this article can be used to every kind of risk.

1. Theory from the literature

The author analyzes the basic operations on random variables. An especially rich theory is associated with the minima and maxima of the stock prices. It is a theory of extreme values. It is a main subject of the dissertation about the futures contracts (Echaust, 2014). According to this author the theory of maxima and minima is especially applied especially for futures because they achieve very high profits and losses in comparison with other financial instruments. Echaust proves that in tails losses are much more correlated than profits. It means that the series of big losses are much more probable for futures derivatives. In this article the author will take into consideration the maximum and minimum function in the theoretical part of the paper.

In the literature only the probability of independent random variables operations is defined. For a sum it is the cumulative function of a sum of random variables which is modeled as a convolution of cumulative functions. Density function of a sum according to some authors is a convolution of density functions (academic lecture – Jurlewicz, 2010).

Now the author will present the known from the literature probability of a product, $Z = XY$. If the random pair (X, Y) has a discrete distribution with the assigned sequence

$\left\{\left(x_n, y_k, p_{nk}\right), n \in T_1, k \in T_2\right\}$, the random variable *Z* has also discrete distribution with a given sequence $\left\{z_j, p\left(z\right)_j\right\}$, one-to-one sequence is created from different numbers of the form $x_n y_n$ and $p_j^{(Z)}$ is a sum of such sequence elements $\left\{p_{nk}\right\}$ with such a numbers n_k for which $z_j = x_n y_k$.

With assumption that P(X = 0) = 0, we may write

$$p_j^{(Z)} = \sum_{n \in T_1} P\left(X = x_n, Y = \frac{z_j}{x_n} \right).$$

With assumption that $P(Y = 0) = 0$, we receive

$$p_j^{(Z)} = \sum_{n \in T_1} P\left(X = \frac{z_j}{y_k}, Y = y_k \right).$$

When the random vector (X, Y) has a density function $f(x, y)$, the random variable Z has a continuous distribution

$$f_Z\left(z\right) = \int_{-\infty}^{\infty} \frac{1}{|x|} f\left(x, \frac{z}{x} \right) dx = \int_{-\infty}^{\infty} \frac{1}{|y|} f\left(\frac{z}{y} . y \right) dy .$$

Density function of a sum is a convolution of density functions.

Now the author will present the probability of a product $Z = X/Y$.

If random pair (X, Y) has a discrete distribution with the assigned sequence $\left\{\left(x_n, y_k, p_{nk}\right), n \in T_1, k \in T_2\right\}$, the random variable *Z* has also a discrete distribution with a given sequence $\left\{z_j, p\left(z\right)_j\right\}$. One-to-one sequence is created from different numbers of the form $x_n y_n$ and p_j^{Z} is a sum of such sequence elements $\left\{p_{nk}\right\}$ with such numbers n_k, for which $z_j = \frac{x_n}{y_k}$.
We can write

$$p_j^{(Z)} = \sum_{n \in T_{21}} P\left(X = z_j y_k, Y = y_k\right).$$

When a random vector (X, Y) has a continuous distribution, the random variable Z has a continuous random distribution with the density function as follow:

$$f_Z\left(z\right) = \int_{-\infty}^{\infty} |y| \, f\left(zy, y\right) dy .$$

The author will also take into consideration the basic random distribution parameters below:

$$E(Z) = E_g(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dF_{X,Y}(x,y),$$

if it exists.

For every random variable X and Y

$$E(X+Y) = E(X) + E(Y).$$

If D^2X and D^2Y exist, we receive:

$$D^2(X+Y) = D^2X + D^2Y + 2\big(E(XY) - E(X)E(Y)\big).$$

X and Y are independent then and only then

$$F_{X,Y}(x,y) = F_X(x)F_Y(y).$$

Then

$$EXY = EXEY$$

and

$$D^2(X+Y) = D^2X + D^2Y.$$

These results come from an academic lecture (Jurlewicz, 2010).

2. Theoretical analysis

First of all, we will conduct the analysis for independent variables. The sum of risk variables may be defined in the following way:

A) The sum of risk variables may be defined in the following way:

$$Z = X + Y = \underset{i,j}{\wedge}\left\{z = x_i + y_j\right\}.$$

For the example placed below the author receives

$$X = \{1,4,5,7\},\ p_i = \left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\} \qquad Y = \{2,3,5,6\},\ p_i = \left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\},$$

$$Z = \{3, 4, 6, 7, 8, 9, 10, 11, 12\}.$$

Suppose that X and Y are independent discrete random variables with the probability functions $P(X = x)$, $P(Y = y)$, then and only then the probability function of random variable $X + Y$ is expressed by the formula:

$P\left(Z=z_k\right)=\sum\left(x=x_i\right)P\left(Y=z_k-x_i\right);\left(z_k=x_i+y_j\right)$ (academic lecture, Jurlewicz, 2010).

$$P\left(Z=3\right)=\frac{1}{20}+\frac{1}{4}\cdot 0+\frac{1}{4}\cdot 0+\frac{1}{4}\cdot 0=\frac{1}{20}$$
$$P\left(Z=4\right)=\frac{1}{20}.$$

Suppose that X and Y are dependent variables. The probability in this case is as follows.

$$P(Z=z_k)=\sum_i \begin{array}{l}\rho\left(x=x_i,y=z_k-x_i\right)\cdot\sqrt{P\left(x=x_i\right)\left(1-P\left(x=x_i\right)\right)P\left(Y=z_k-x_i\right)\left(1-P\left(z_k-x_i\right)\right)}\\+P\left(x=x_i\right)P\left(Y=z_k-x_i\right)\end{array}$$

(the author's own formula).

For the product of random variable by the arbitrary number λ

$$Z=\lambda X,$$

$X=\{1,4,5,7\}$, $p_i=\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\}$ (the author's own example),

$$Z=\lambda X=\{\lambda 1,\lambda 4,\lambda 5,\lambda 7\}.$$

Multiplying loss by the constant different than zero will cause that the loss will enlarge proportionally in relation with its value at the beginning. We can't say anything about the probability of such case.

B) The median of a sum of random variables. The first case.

$$Z=Med(X+Y).$$

We rank the values of the both random variables. Then we calculate the median of the so received series.

The example may be the following.

$$X=\{1,4,5,7\},\ p_i=\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},\qquad Y=\{2,3,5,6\},\ p_i=\left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\},$$

$Med=Med\{1,2,3,4,5,6,7\}=4$ (own definitione and example).

C) The median of a sum of random variables, the second way of calculation (the author's own definition).

$$Z=Med(X+Y)=Med\left(\bigwedge_{i,j}\left(x_i+y_j\right)\right),$$

$$X = \{1,4,5,7\}, \quad p_i = \left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},$$

$$Y = \{2,3,5,6\}, \quad p_i = \left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\},$$

$$Z = \{3, 4, 6, 7, 8, 9, 10, 11, 12\},$$

$$Med(X) = 8.$$

The median of a sum of random variables may be calculated as a median of a random variable which denotes a sum of losses which may appear with the given probability. (It is the author's own definition)

D) A product of random variables

$$Z = X \times Y.$$

The probability for independent variables X and Y is the following.

$$P(Z = z_k) = P(X = x_i)P(Y = y_j).$$

The probability for dependent variables X and Y is more complicated, as below.

$$P(Z = z_k) = \rho(x_i, y_j) \times \sqrt{P(x_i)(1-P(x_i))P(y_j)(1-P(y_j))} + P(X = x_i)P(Y = y_j)$$

(Kubiak, 1980).

In practice one calculates the product as a combination.

Consider the following example:

$$X = \{1,4,5,7\}, \quad p_i = \left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},$$

$$Y = \{2,3,5,6\}, \quad p_i = \left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\}.$$

Then:

$$X \times Y = \{1 \times 2, 1 \times 3, 1 \times 5, 1 \times 6, 4 \times 2, 4 \times 3, 4 \times 5, 4 \times 6, 5 \times 2, 5 \times 3, 5 \times 5, 5 \times 6, 7 \times 2, 7 \times 3, 7 \times 5, 7 \times 6\}.$$

The obvious consequence for a product are the formulas for the variance below:

$$D^2(X \cdot Y) = E(XY)^2 - (E(XY))^2.$$

for an independent case

$$D^2(X\cdot Y)= E(XY)^2-(E(XY))^2=E\left(X^2Y^2\right)-\left(E(X)E(Y)\right)^2 \text{ (the author's own formula).}$$

for a dependent case

$$D^2(X\cdot Y)=E(XY)^2-(E(XY))^2 \text{ (the author's own formula).}$$

E) Division of random variables

$$Z = X/Z.$$

The probability is achieved with the assumption that both random events will occur.

The probability for independent variables X and Y.

$$P(Z=z_k)=P(X=x_i)P(Y=y_j).$$

The probability for dependent variables X and Y.

$$P(Z=z_k)=\rho(x_i,y_j)\cdot\sqrt{P(x_i)(1-P(x_i))P(y_j)(1-P(y_j))}+P(X=x_i)P(Y=y_j)$$

(the author's own formula).

Let's take into consideration the following example

$$X=\{1,4,5,7\},\ \ p_i=\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},$$

$$Y=\{2,3,5,6\},\ \ p_i=\left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\}.$$

Then the author receives:

$$X/Y=\{1/2,1/3,1/5,1/6,4/2,4/3,4/5,4/6,5/2,5\cdot/3,5/5,5/6,7/2,7/3,7/5,7/6\}.$$

For the division, the basic statistic is:

$$D^2(X/Y)=E(X/Y)^2-(E(X/Y))^2 \text{ (the author's own formula)}$$

for an independent case:

$$E(X/Y)^2-(E(X/Y))^2=E\left(X^2/Y^2\right)-\left(E(X)/E(Y)\right)^2 \text{ (the author's own formula).}$$

F) The maximum of random variables may be defined in two ways:

$$Z = Max(X, Y).$$

Suppose that:

$$X = \{1,4,5,7\}, \quad p_i = \left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},$$

$$Y = \{2,3,5,6\}, \quad p_i = \left\{\frac{1}{5},\frac{2}{5},\frac{1}{5},\frac{1}{5}\right\}.$$

The author receives the following combinations:

{(1,2), (1,3), (1,5), (1,6), (4,2), (4,3), (4,5), (4,6), (5,2),(5,3), (5,5), (5,6),
(7,2), (7,3), (7,5), (7,6)}.

The interpretation in the area of risk management is that it will realize the bigger value of risk with certain probability.

$$Max(X, Y) = \bigwedge_{i,j} Max(x_i, y_j).$$

We may assume that we rank all the values of the random variable and then we appoint the maximum

For the above example

{1, 4, 5, 7, 2, 3, 5, 6} maximum. $Max = 7$.

The probability of the received value

$$FZ(z) = FX(z)FY(z).$$

The proof for cumulative function is the following:

$$FZ(z) = P(\max(X,Y) < z) = P(X < z, Y < z) =$$

$$= \text{independence} = P\left(X < z\right)P\left(Y < z\right) = FX(z)FY(z) \text{ (academic lecture, Jurlewicz, 2010).}$$

Now the author will perform the analysis for dependent variables.

She receives the following formula:

$$FZ(z) = P(\max(X,Y) < z) = P(X < z, Y < z) =$$

$$= \text{dependent} = \rho\left(X,Y\right)\sqrt{F\left(X\right)(1-F\left(X\right)F\left(Y\right)\left(1-F\left(Y\right)\right)} + F(X)F(Y)$$

(the author’s own formula).

G) The minimum of random variables

$$Z = min(X, Y).$$

We define the difference analogically to the maximum.

Suppose that X and Y are independent random variables respectively with the distribution functions $FX(x)$ i $FY(y)$.

Then the random variable has a distribution function as follows:

$$FZ(z) = 1-(1-FX(z))(1-FY(z)).$$

Justification

$$FZ(z) = P(\min(X,Y) < z) = 1-P(\min(X,Y) < z),$$

$$1-P(\min(X,Y)-z) = 1-P(X-z, Y-z) = 1 - P(min(X, Y)-z) = 1 - P(X-z, Y-z) =$$

$$= \text{independent} = 1-P(X-z)P(Y-z) = 1-(1-FX(z))(1-FY(z))$$

(academic lecture, Jurlewicz, 2010).

Now the author will perform the analysis for dependent variables.

$$FZ(z) = P(\min(X,Y) < z) = 1-P(\min(X,Y)-z) = 1-P(X-z, Y-z),$$

$$\text{dependent} = 1 - \rho\left(X-z, Y-z\right)\sqrt{P\left(X-z\right)(1-P\left(X-z\right)P\left(Y-z\right)\left(1-P\left(Y-z\right)\right)} +$$

$$+ P(X-z)P(Y-z) = 1-P(X-z)P(Y-z) =$$

$$= 1-\rho\left(X-z, Y-z\right)\sqrt{\left(1-FX\left(z\right)\right)\left(1-\left(1-FX\left(z\right)\right)\right)FY\left(z\right)\left(1-\left(1-FY\left(z\right)\right)\right)} +$$

$$+\left(1-FX\left(z\right)\right)\left(1-FY\left(z\right)\right) \text{ (the author's own formula).}$$

3. Interpretations of the operations

The presented operations may be used in many situations of risk, and are applicable to many kinds of risk. The sum of risk variables may be applied when we take into consideration a portfolio of assets, for example a financial portfolio, and we are interested how big a sum of losses or a sum of profits is. It may be a portfolio of financial instruments or, for example, credits. We are also interested what is the probability of the total loss or profits of the portfolio. Sometimes there may be a correlation between losses from different assets. In this situation the probability of loss of the portfolio is different than for independent variables.

When we take into consideration the carambola, a flood or other mass catastrophe, the loss at the beginning will be multiplied, but not necessarily proportionally to the loss at the beginning. We cannot say anything about probability in this case except that it is usually small.

A product of risk variables has a similar interpretation. When we multiply the risk of a financial instrument, such as playing on the stock market with a lever, then our loss or win will be multiplied proportionally. Another example is a pyramid scheme.

The division of risk variables may be a model of a lotto game. The more people play successfully, the more the win is divided. So, we have two kinds of risk – the risk of a sum of money to win and the risk of the quantity of people who have chosen good numbers. The probability of the win is a product of probabilities of the sum of money and the people who guessed the correct numbers. In this game we have to deal with the dependent random variables. There exists the correlation between the size of accumulation and the number of people who are the winners. The biggest is the accumulation, the more people play.

In some situations, we are interested in the calculation of a median of a sum of variables. For example, when we calculate the risk of financial portfolio and we want to assess the risk with the use of median. The median is a one of the best measures of risk because it fulfills all axioms of a coherent measure of risk (Buszkowska, 2015). The median of a sum is analyzed when we check the conditions of the coherent measure of risk for the median.

The maximum loss may be a model of fluctuations on the Shepard card in production management (Hamrol, 2007). The manager is interested only with the biggest fluctuations which border the border line and mean the production process is unstable. Some of the samples are very different than others because of some feature. The reason may be, for example, that the machine has broken down or the operator does not work well. Another example is stock investing with take profit option. If a trader uses take profit, they will be interested only with maximum wins which border a special limit. On the other hand, they are limited with the deposit on the account or stop loss. When they play with a small deposit, they may be interested with the minimum fluctuations, which denotes the profit for them.

Conclusions

In the article the author presents and discusses different operations on risk variables which are a special interpretation of random variables. She achieves different formulas of probability for these random variables in the dependent case, for the max, min, fraction and sum of random variables. She calculates the formulas for the expected values and the distribution function for some operations of random variables. She gives two propositions of defining the median of a sum of variables. At the end the author proposes some interpretations of the operations in the field of economics and production management.

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