*Folia Oeconomica Stetinensia*
DOI: 10.2478/v10031-011-0036-8

# GOODNESS OF FIT TESTS IN MODELING THE DISTRIBUTION OF THE DAILY RATE OF RETURN OF THE WIG20 COMPANIES

Kamila Bednarz, MA

*Department of Quantitative Methods*
*Faculty of Management and Economics of Services*
*University of Szczecin*
*Cukrowa 8, 71-004 Szczecin*
*e-mail: kamila.bednarz@wzieu.pl*

**Received 9 September 2011, Accepted 23 December 2011**

## Abstract

In this paper a classic rate of return was examined. Due to a limited quantitative range, the study included only the modeling of the rate of return distribution of the WIG20 index and its companies by means of the Laplace distribution and the Gaussian distribution. Additionally, the goodness of fit tests and methods of estimating the aforementioned distributions parameters were thoroughly covered. When applying the Laplace distribution to modeling the rate of return distribution the parameters were determined by means of two methods: the method of moments and the maximum likelihood method. The maximum period was determined, for which usefulness of the distribution in modeling the rates of return distribution was observed, as well as the results of the chi-square test for class intervals with varying length ensuring equal probability, and for intervals with identical length considering two methods of determining the theoretical size: in accordance with the cumulative distribution function as well as on the basis of the probability density function.

**Keywords:** goodness of fit tests, Kolmogorov test, chi-square test, class intervals.

**JEL classification:** C12, E43.

## Introduction

One of the most important financial tools of a privately-owned nature is a stock. It is a type of security that signifies ownership in a corporation. A basic concept connected with stocks and used in virtually all methods of analysis is a rate of return (ROR). It chiefly enables profitability evaluation of investment in stocks.

The rate of return is treated as a random variable, the distribution of which is obtained through modeling of a rate of return distribution. In the modeling, the most frequently applied distributions are: the normal distribution, GED distribution, alpha-stable distribution and Student's t-distribution.

This paper concentrates only on modeling the rate of return distribution of the WIG20 index and its companies by means of the Laplace distribution and the Gaussian distribution.

The objective of this paper was to compare the results of the chi-square test for class intervals of different forms. Taken into account were class intervals that guarantee identical theoretical number of elements and class intervals with identical length. In the latter case two methods of determining the theoretical number of elements were considered: the one based on the value of the cumulative distribution function and another method using the value of the probability density function in the middle of the class interval.

## 1. Estimation of distribution parameters

There are two most popular methods of estimation of distribution parameters: the method of moments and the maximum likelihood method[1].

In the case of a normal distribution described by the following probability density function:

$$fn(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

both methods lead to identical dependencies[2]:

$$\begin{gathered} \hat{\mu} = \overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \\ \hat{\sigma}^2 = S^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2 \end{gathered} \tag{2}$$

In the literature[3] it has been shown that the estimator $S^2$ is biased and it is recommended to use an unbiased estimator of the variance:

$$S_1^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{3}$$

Yet another approach[4], where the accepted criterion was that of minimization of the mean square error of the variance estimation obtaining:

$$S_2^2 = \frac{1}{N+1}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{4}$$

In this paper the estimator of variance described by formula (2) will be applied, which is a compromise between dependencies (3) and (4).

Estimation of the Laplace distribution parameters given by the probability density function[5]:

$$fL(x,\mu,\lambda) = \frac{\lambda}{2}\exp(-\lambda|x-\mu|) \tag{5}$$

The method of moments yields the following estimations:

$$\begin{gathered}\hat{\mu} = \overline{x}\\ \hat{\lambda} = \sqrt{\frac{2}{S^2}}\end{gathered} \tag{6}$$

where $\overline{x}$ and $S^2$ are described by equation (2).

As a result of using the maximum likelihood method, we obtain:

$$\begin{gathered}\hat{\mu} = median\ (x_i)\\ \hat{\lambda} = \frac{N}{\sum_{i=1}^{N}|x_i - \hat{\mu}|}\end{gathered} \tag{7}$$

## 2. Goodness of fit tests

### 2.1. $\chi^2$ test

$\chi^2$ goodness of fit test (Pearson, 1900) can be applied to both a discrete and linear random variable.

In the literature the following formulas binding the number of classes $L$ and the number of observations $N$ can be found:

$$L = 5\log N \tag{8a}$$

$$L = \sqrt{N} \tag{8b}$$

$$L = 3{,}764(N-1)^{0{,}4} \tag{8c}$$

Another approach can be found[6], where they provide a chart binding the number of classes $L$ with the number of observations $N$. Having compared the data included in the chart with the results of equations (8a–8c), it can be claimed that equation (8b) yields the values closest to the data in the chart.

The results of the $\chi^2$ test are presented in the form of a normalized statistic value:

$$h = \frac{H}{Hkr} \tag{9}$$

where: Hkr value taken from the $\chi^2$ distribution table for $r = L - 3$ degrees of freedom at a significance level of $\alpha = 0.05$.

The hypothesis that the empirical distribution fits the assumed theoretical distribution was rejected since h >1.

The literature[7] recommends application of class intervals that guarantee identical theoretical number of elements, which leads to the greater power of a test[8]. In this case the values of class interval limits $y_i$ (0, 1, ..., $L$) are determined using the following formula:

$$y_j = \begin{cases} \mu + \dfrac{1}{\lambda}\ln\left(\dfrac{2j}{L} + 10^{-5}\right) & for \;\; j < \dfrac{L}{2} \\ \mu - \dfrac{1}{\lambda}\ln\left[2\left(1 - \dfrac{j}{L}\right) + 10^{-5}\right] & for \;\; \dfrac{L}{2} \le j \le L \end{cases} \tag{10}$$

However the $\chi^2$ test is commonly applied to class intervals of the same length $c$[9]:

$$c = \frac{M_1 - m_1}{L} \tag{11}$$

where:

$m_1 = \min(x_i)$,

$M_1 = \max(x_i)$.

For this case the values of class interval limits $y_i$ (0, 1, ..., $L$) are determined using the following formula:

$$y_1 = m_1 + ci \tag{12}$$

where $c$ is described by equation (11), i = 0, 1, ..., $L$.

Values of the theoretical number of elements $nt_j$ for individual intervals are determined using the following formula[10]:

$$nt_j = N(F(y_j) - F(y_{j-1})) \tag{13}$$

where: $F(y)$ the cumulative distribution function $j = 1, 2, ..., L$.

For the Laplace distribution the following formula holds:

$$nt_j = \frac{N}{2}\begin{cases} (1-e^{-\lambda c})e^{\lambda(y_j-\mu)} & for \;\; j \le \dfrac{L}{2} \\ (e^{\lambda c}-1)e^{-\lambda(y_j-\mu)} & for \;\; \dfrac{L}{2} \le j \le L \end{cases} \tag{14}$$

In the case of the Gaussian distribution an analytic form $nt_j$ cannot be determined as is the case for the Laplace distribution (equation (14)), hence one has to make do with equation (13) reading the values of the cumulative distribution function in statistical tables.

Considering this inconvenience many authors, among others[11], suggest determining the theoretical number of elements on the basis of the probability density function:

$$nt_j = N\,c\,f(y'_j) \tag{15}$$

where:

$f(y)$ – probability density function;

$y'_j$ – mid-value of the class interval (class mark),

$c$ – described by (11).

From equations (1), (5), (15) the following forms of the theoretical number of elements can be obtained:

– for the Gaussian distribution:

$$nt_j = \frac{N\,c}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(y'_j-\mu\right)^2}{2\sigma^2}\right) \tag{16}$$

– for the Laplace distribution:

$$nt_j = N\,c\,\frac{\lambda}{2}\exp\left(-\lambda\left|y'_j-\mu\right|\right) \tag{17}$$

## 2.2. The Kolmogorov test

The test is applied to verify the hypothesis that a random variable $X$ of a linear type has a cumulative distribution function $F(x)$. A critical value at a significance level of $\alpha = 0.05$ is derived from the formula:

$$Dkr = \frac{1,354}{\sqrt{N}} \quad for \quad N \ge 100 \tag{18}$$

In the paper the normalized value of the statistic $K$ is applied:

$$K = \frac{D}{Dkr} \tag{19}$$

where: D value of the statistic determined in the Kolmogorov test.

If $K > 1$ then the hypothesis that the population under study has the cumulative distribution function $F(x)$ should be rejected.

The Kolmogorov test relates to a theoretical cumulative distribution function F(x) with known parameters. If the parameters, on which the theoretical cumulative distribution function depends, are not known and we estimate them based on a sample, then the results of the Kolmogorov test should be treated with reserve[12]. Despite these reservations the Kolmogorov test is commonly applied and will be used in this paper as well.

## 3. Results of modeling the rate of return distribution of the WIG20 companies

In the paper closing stock prices (daily data) of the WIG20 index and the companies included in the index on 30.09.2010 were considered. The period of observation taken into account included quarters counted backwards from the given date for which positive results of the Kolmogorov and $\chi^2$ tests were obtained in relation to the modeling of ROR distribution of this index and the companies using the Laplace and normal distributions. As the number of observations increased, a substantial deterioration in modeling results could be observed.

The last two columns in the table below (Table 1) determine the maximum applicability period of a given distribution to ROR modeling. We can observe that in only one case (PEKAO, Gaussian distribution) the maximum period was determined by the Kolmogorov test. And in all other cases it was the chi-square test that determined the length of the period of distribution applicability. Therefore it can be concluded that when conducting a goodness of fit test, the chi-square test alone is sufficient.

Table 2 presents the results of modeling of the ROR distribution of the WIG20 companies and the WIG20 index itself by means of the normal distribution. It shows the maximum applicability period of the distribution as well as the results of the chi-square test for unequal class intervals ensuring equal probability (*hpn*), and for intervals of the same length including two methods of determining the theoretical size: in accordance with the cumulative distribution function (*hn*1) as well as on the basis of the probability density function (*hn*2). The positive result of the test occurs when at least one value of a normalized statistic of the chi-square test (*hpn*, *hn*1, *hn*2) is smaller than 1.

Table 1. Results of modeling of ROR distribution (daily data) with the use of the Laplace and Gaussian distributions

| Company name | Number of quarters leading to a positive test result | | | | Maximum number of quarters | |
|---|---|---|---|---|---|---|
| | Laplace distribution | | Gaussian distribution | | Laplace distribution | Gaussian distribution |
| | Kolmogorov test | Chi-square test | Kolmogorov test | Chi-square test | | |
| ASSECOPOL | 1–10 | 1–10 | 1–4 | 1 | 10 | 1 |
| BRE | 1–4 | 1–2; 4 | 1–4 | 1 | 4 | 1 |
| BZWK | 1–12 | 1–12 | 0 | 0 | 12 | 0 |
| CEZ | 1–12 | 1–5; 12 | 1–6 | 1–4; 6 | 12 | 6 |
| CYFRPLSAT | 1–6 | 1–4; 6 | 1–6 | 2–3 | 6 | 3 |
| GETIN | 1–4 | 1–2; 4 | 1–4 | 1; 4 | 4 | 4 |
| GTC | 1–5 | 1–5 | 1–4 | 1 | 5 | 1 |
| KGHM | 1–13 | 1–9 | 1–7 | 1–7 | 9 | 7 |
| LOTOS | 1–9 | 1–9 | 1–9 | 1–2 | 9 | 2 |
| PBG | 1–12 | 1–12 | 1–3 | 1 | 12 | 1 |
| PEKAO | 1–10 | 1–9 | 1–5 | 1–6 | 9 | 5 |
| PGE* | 1–3 | 1–3 | 1–3 | 1; 3 | 3 | 3 |
| PGNIG | 1–8; 16 | 1–4; 12–16 | 1–3 | 1; 3 | 16 | 3 |
| PKNORLEN | 1–9 | 1–8 | 1–9 | 1–9 | 8 | 9 |
| PKOBP | 1–14 | 1–14 | 1–4; 12 | 1–3 | 14 | 3 |
| POLIMEXMS | 1–6 | 1–6 | 1–2 | 2 | 6 | 2 |
| PZU** | 100 days | 100 days | 100 days | 100 days | 100 days | 100 days |
| TPSA | 1–10 | 2–3; 5–6; 8–10 | 1–7 | 1 | 10 | 1 |
| TVN | 1–22 | 1–22 | 1–22 | 1–2 | 22 | 2 |
| WIG20 | 1–36 | 1–16; 24–36 | 1–4 | 1–3 | 36 | 3 |

* Data from 6.11.2009 – included the maximum of three quarters.
** Data from 12.05.2010 – included the maximum of 100 days.

Source: Author's own study.

Table 2. Results of the chi-square test for the normal distribution

| Company name | Number of quarters | *hpn* | *hn*1 | *hn*2 |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| ASSECOPOL | 1 | 0.56 | 0.904 | 0.942 |
| BRE | 1 | 0.591 | 2814.92 | 5027.21 |
| CEZ | 6 | 0.917 | 48286 | 67297 |
| CYFRPOLSAT | 3 | 0.921 | 1.684 | 1.75 |
| GETIN | 4 | 1.464 | 0.981 | 0.991 |
| GTC | 1 | 1.831 | 0.054 | 0.051 |
| KGHM | 7 | 0.933 | 0.957 | 0.97 |
| LOTOS | 2 | 0.447 | 0.658 | 0.673 |
| PBG | 1 | 0.797 | 0.239 | 0.304 |
| PEKAO | 6 | 0.744 | 8.34 | 8.993 |
| PGE | 3 | 0.712 | 1.034 | 1.045 |

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| PGNIG | 3 | 0.876 | 286.62 | 401.68 |
| PKNORLEN | 9 | 0.792 | 18.3 | 20.35 |
| PKOBP | 3 | 0.712 | 0.529 | 0.534 |
| POLIMEX | 2 | 1.965 | 0.925 | 0.949 |
| PZU | 1,5 | 0.327 | 10.069 | 14.02 |
| TPSA | 1 | 1.228 | 0.741 | 0.764 |
| TVN | 2 | 0.735 | 25.07 | 31.99 |
| WIG20 | 3 | 0.817 | 1.422 | 1.512 |

Source: Author's own study.

Companies BIOTON and BZWK were excluded from the table since their normalized statistics were larger than 1 for any given period. Taking into account the fact that, at present, these companies are not included in the WIG20 index the following hypothesis can be put forward: problems with the modeling of the ROR distribution can be a warning signal for people interested in buying stocks.

While comparing the values of normalized statistics $hn1$ and $hn2$ it can be noticed that in only one case (GTC) the following inequality holds $hn2 < hn1$. However for all other cases included in the table $hn1 < hn2$, which means that determining the theoretical size in the chi-square test on the basis of the cumulative distribution function yields better results than on the basis of the probability density function.

In the literature it is recommended to use class intervals ensuring identical theoretical number of elements, which leads to the greater power of a test. This recommendation can be confirmed by the data in Table 2, where for 13 companies holds $hpn < hn1$.

Table 3. Results of the chi-square test for the Laplace distribution

| Company name | Number of quarters | *hpL* | hL1 | *hL2* |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| ASSECOPOL | 10 | 2.007 | 0.966 | 0.96 |
| BRE | 4 | 1.346 | 0.803 | 1.026 |
| BZWK | 12 | 1.675 | 0.978 | 0.974 |
| CEZ | 12 | 0.996 | 1.455 | 1.47 |
| CYFRPOLSAT | 6 | 1.924 | 0.824 | 0.86 |
| GETIN | 4 | 1.306 | 0.763 | 0.793 |
| GTC | 5 | 0.826 | 1.014 | 1.099 |
| KGHM | 9 | 0.9 | 0.918 | 1.123 |
| LOTOS | 9 | 0.988 | 2.363 | 2.559 |
| PBG | 12 | 2.42 | 0.967 | 0.968 |
| PEKAO | 9 | 0.962 | 0.82 | 1.001 |
| PGE | 3 | 0.615 | 1.054 | 1.087 |
| PGNIG | 16 | 6.274 | 0.594 | 0.626 |

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| PKNORLEN | 8 | 1.146 | 0.94 | 0.96 |
| PKOBP | 14 | 1.118 | 0.907 | 0.951 |
| POLIMEX | 6 | 1.996 | 0.974 | 0.982 |
| PZU | 1,5 | 0.441 | 0.659 | 0.779 |
| TPSA | 10 | 1.436 | 0.562 | 0.604 |
| TVN | 22 | 2.956 | 0.929 | 0.986 |
| WIG20 | 36 | 1.208 | 0.766 | 0.86 |

Source: Author's own study.

Table 3 comprises the results of the chi-square test for the modeling of the ROR distribution of the WIG20 companies by means of the Laplace distribution. The following columns comprise the values of the normalized statistic values of the $\chi^2$ test: $hpL$ – unequal class intervals; $hL1$ ($hL2$) – class intervals of the same length and the same number of elements determined on the basis of the cumulative distribution function ($hL2$ – probability density function).

The results for BIOTON were excluded from the table since the normalized statistics for the company were larger than one for any given period.

The analysis of the table content proves a definite advantage (as many as 19 cases) of the estimation of the theoretical number of elements on the basis of the cumulative distribution function – it was only BZWK company for which the inequality held $hn2 < hn1$. Application of the probability density function to determine the theoretical number of elements increases statistic values ($hn2 > hn1$) but considerably simplifies calculations.

In contrast with Table 2, the data concerning unequal class intervals presented in Table 3 looks different. Namely, according to Table 3, in 14 cases (out of 20) the following inequality holds $hn1 < hpn$, which confirms the advantage of class intervals with the same length. Hence, for the Laplace distribution the situation is opposite to that of the Gaussian distribution, for which class intervals with unequal length were preferable.

## Conclusions

In the paper selected aspects of applying goodness of fit tests were considered. One of them was the choice of a proper formula for estimation of distribution parameters. While applying the Laplace distribution to the modeling of a rate of return, the parameters were determined using two methods: the method of moments (equation (6)) and the maximum likelihood method (equation (7)), and consequently the values of the chi-square test statistic were determined. Based on the calculations it was concluded that the method of moments yielded larger values of the normalized statistic than the maximum likelihood method. As a result, all the results presented in Table 3 refer to the Laplace distribution parameters determined using the maximum likelihood method.

As far as the Gaussian distribution is concerned, such a problem does not exist – both methods lead to the same dependencies (equation (2)).

Another conclusion drawn on the basis of the data in Table 1 is that the chi-square test turned out to be much more 'demanding' – when determining the maximum applicability period of the distribution to modeling – than the Kolmogorov test. Therefore it is recommended to use the $\chi^2$ test with the exclusion of the Kolmogorov test.

With reference to the objective of this paper provided in Introduction, the issue of class intervals of the same length has been definitely resolved – for both the normal and Laplace distributions. Namely, it is advisable to determine the theoretical number of elements on the basis of the cumulative distribution function (equations (13) and (14)) – smaller value of the statistic compared to the method based on the probability density function.

In the case of unequal class intervals ensuring equal theoretical number of elements, no definite answer has been found. This method turned out to be better for the Gaussian distribution – yielding a smaller value of a statistic – than for intervals of the same length. The opposite holds true for the Laplace distribution for which intervals of the same length are recommended.

Summing up the conclusions, it is advisable to conduct the $\chi^2$ test for both unequal intervals (equation (11)) and intervals of the same length (equation (15)) and to choose as a result the smallest value of the test.

## Notes

[1] Sobczyk (2004), p. 143.

[2] Fisz (1969), p. 460.

[3] Ibidem p. 481.

[4] Krzyśko (1997), p. 32.

[5] Purczyński (2003), p. 135.

[6] Krysicki, Bartos, Dyczka, Królikowska, Wasilewski (1995), p. 110, Domański, Pruska (2000), p. 170.

[7] Fisz (1969), p. 457, Krysicki, Bartos, Dyczka, Królikowska, Wasilewski (1995), pp. 100–111.

[8] Krysicki, Bartos, Dyczka, Królikowska, Wasilewski (1995), p. 111.

[9] Tarczyński (2002), p. 48.

[10] Krysicki, Bartos, Dyczka, Królikowska, Wasilewski (1995), p. 104, Domański, Pruska (2000), p. 168.

[11] Tarczyński (2002), p. 48, Tarczyński, Mojsiewicz (2001), p. 55.

[12] Domański, Pruska (2000), p. 171, Fisz (1969), p. 463.

## References

Domański, Cz., Pruska, K. (2000). *Nieklasyczne metody statystyczne*. Warszawa: PWE.

Fisz, M. (1969). *Rachunek prawdopodobieństwa i statystyka matematyczna*. Warszawa: PWN.

Krysicki, W., Bartos, J., Dyczka, W., Królikowska, K., Wasilewski, M. (1995). *Rachunek prawdopodobieństwa i statystyka matematyczna w zadaniach*. Cz. II. *Statystyka matematyczna*. Warszawa: PWN.

Krzyśko, M. (1997). *Statystyka matematyczna*. Cz. II. Poznań: UAM.

Purczyński, J. (2003). *Wykorzystanie symulacji komputerowych w estymacji wybranych modeli ekonometrycznych i statystycznych*. Szczecin: Uniwersytet Szczeciński.

Sobczyk, M. (2004). *Statystyka*. Warszawa: PWN.

Tarczyński, W. (2002). *Fundamentalny portfel papierów wartościowych*. Warszawa: PWE.

Tarczyński, W., Mojsiewicz, M. (2001): *Zarządzanie ryzykiem*. Warszawa: PWE.