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Supervised Machine Learning with Control Variates for American Option Pricing

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Abstract. In this paper, we make use of a Bayesian (supervised learning) approach in pricing American options via Monte Carlo simulations. We first present Gaussian process regression (Kriging) approach for American options pricing and compare its performance in estimating the continuation value with the Longstaff and Schwartz algorithm. Secondly, we explore the control variates technique in combination with Kriging to further improve the estimation of the continuation value. This method allows to reduce dramatically the standard errors and to improve the stability of the Kriging approach. For illustrative purposes, we use American put options on a stock whose dynamics is given by Heston model, and use European options on the same stock as control variates.

Key Words: American options, Monte Carlo, Gaussian processes, Kriging, LSM, supervised learning, Heston Model, control variates

1. Introduction

American options pricing efficiency remains a topic of heated interest and research in Computational Mathematics and Finance. American style options offer great flexibility in all financial and trading markets such as stock, equity, commodity, credit and forex due to the possibility of an early exercise. However, there are no closed-form analytical valuation of these types of derivatives because of the optimal exercise problem created by the early exercise.

The increasing difficulty of pricing American options when the underlying follows a complex stochastic process dynamics have turned researchers to the use of the Monte Carlo simulations. Monte-Carlo simulations complexity increases linearly with

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the dimension of the problem, whereas methods like finite difference's complexity is exponential. The Monte Carlo approach to price American options is accompanied with the question on how to estimate the continuation value and execute the optimal exercise. Most of the developed methods in this area are related to supervised learning i.e. estimating '$\mathbb{E}[Y|X] = f(X) + \epsilon$' via parametric/non-parametric techniques. Many researcher attempted to solve the problem such as Tsitsiklis and Van Roy [15] with parametric regression, Pizzi and Pellizzari [11] with Nadaraya-Watson kernel regression and Li, Szepesvari and Schuurmans [7] with LSPI algorithm. A major breakthrough in pricing American options with Monte Carlo approach was developed by Longstaff and Schwartz [8] and it is considered a "golden standard" in the industry and literature. Longstaff and Schwartz method (LSM) uses parametric regression onto basis functions. In this paper, the authors have proved that when increasing the number of Monte-Carlo simulations and the basis functions (e.g. polynomial type basis function) the LSM method converges. However, it has slow convergence rate and the computational time increases exponentially as long as the number of basis functions and dimension complexity increase.

In this paper, we review a supervised learning approach namely Gaussian process regression (GPR) which was used the first time by Ludkovski [10] in estimating the continuation value of American style options. Gaussian process regression, also named Kriging, with squared exponential kernel requires only one basis function per dimension to be able to learn the continuation value, creating a smoother exercise boundary. We aim to improve it with the application of control variate and give Kriging more flexibility in pricing American options over the LSM methodology on multidimensional stochastic processes. Also, this will allow Kriging method to learn the exercise policies from much smaller sample and can be applied to the rest of the paths, 'sub-sample strategy' Gramacy and Ludkovski [5].

Our result are mostly focus on the Heston model, a 2-dimensional process, to show the efficiency of the Kriging with control variate over the LSM method. Heston model offers more flexibility in pricing options on the stock market due to its stochastic volatility which can replicate the implied volatility smile observed of the options traded in financial markets.

The paper is structured as follows. In the section 2, we review the backward dynamics principle, LSM approach and Kriging algorithm. We explain how to apply the control variates technique with the aid of the GPR. Section 3 contains our numerical analysis where we test our approach on synthetic and market data. Finally, we summarize our findings in the section 4.

2. Pricing American Options

We assume that the financial market has an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ with finite time horizon $[0, T]$ and $\mathcal{F}_T = \mathcal{F}$. Under the no-arbitrage condition, we can suppose the existence of equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$. Furthermore, let S_t be the state variable (i.e. price of the asset) and the $h(S_t)$ the payoff of the options restricted to the square-integrable space $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

In the pricing framework, we will be using Heston 2-dimensional model to depart from Black-Scholes 1-dimensional dynamics and to show the performance the Kriging method on a multidimensional asset. Heston model is a mathematical model which assumes a non-constant volatility of an underlying asset and was first introduced by Heston [6]. Let S_t be the underlying asset under risk neutral measure with variance v_t that follows a CIR process:

$$\begin{aligned} d\ln S_t &= (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_{1,t}^Q \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t}^Q \\ < dW_{1,t}^Q, dW_{2,t}^Q > &= \rho dt \end{aligned} \tag{1}$$

where $S_0 \geq 0, v_0 > 0$ are initial value of the asset and its variance and $W_{1,t}^Q$, $W_{2,t}^Q$ are standard Wiener process with the following parameters: r- risk free rate, $|\rho| < 1$-correlation of $W_{1,t}^Q$ and $W_{2,t}^Q$, $\kappa > 0$- mean reverting rate of variance, $\theta > 0$- long run variance, and $\sigma > 0$- volatility of variance.

American options with maturity T can be formulated as a process $\{H_\tau \in \mathcal{F}_\tau\}_{0\leq\tau\leq T}$ which represents the discounted payoff of the option in case of exercise at time τ. Denote $\mathcal{T}$ of admissible stopping time τ taking values $[0,T]$. Then the initial price of this option can be written as:

$$P_0 = \sup_{\tau\in\mathcal{T}} \mathbb{E}[H_\tau].$$

Let the financial asset S_τ follow Heston dynamics then an American put option with maturity T and strike price K written on an asset S_τ has payoff $h(S_\tau) = (K - S_\tau)^+$ and the following price at time t:

$$P(S, v, t) = \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+] \tag{2}$$

The supremum is achieved for stopping optimal time τ^*:

$$\tau^* = \inf\{\tau \in \mathcal{T} : S_\tau \leq B_\tau\} \tag{3}$$

where B_τ is called optimal exercise boundary.

2.1. Monte Carlo Formulation

In order to be able to tackle numerically the American option pricing problem with Monte Carlo method, we have to depart from the continuous case and restrict to option classes with discrete exercise. Bermudan options are a hybrid between American options and European which allows the option to be exercised at specific dates before maturity. In our case lets define uniform discretization of the time interval $[0,T]$ with distance $\Delta t = \frac{T}{N}$ and $0 = t_0 < t_1 < ... < t_N = T$ equidistant exercise opportunities. The asset value S_t is also simulated at the same time steps $\{t_i\}_{i\in\{0,..,N\}}$ as the discretization. The simulation of $S_{t_1}, ..., S_{t_N}$ is affected by discretization error which is described in detail by Kloeden & Platen [9] and Glasserman [3].

Monte Carlo methods for pricing American option are based on the backward dynamic programming formulation which estimates recursively the value of the option. Let's denote $X = (S, v)$ the underlying stochastic process, $h(X_{t_i})$ the discounted payoff of the option, and V_{t_i} the discounted value of the option at time t_i. The recursive estimation of the option value is defined in the following way:

$$\begin{aligned} V_{t_N} &= h(X_{t_N}) \\ V_{t_{i-1}} &= \max\{h(X_{t_{i-1}}), \mathbb{E}[V_{t_i}|X_{t_{i-1}}]\} \\ & i = 1, 2, ..., N \end{aligned} \tag{4}$$

The above equation states that the value of the option at expiration time t_N is exactly the payoff of the option $h(X_{t_N})$. At any other time $0 \leq t_i < t_N$, the value of the option $\bar{V}_{t_{i-1}}$ is the maximum between immediate exercise $\bar{h}(X_{t_{i-1}})$ and the continuation value $\mathbb{E}[V_{t_i}|X_{t_{i-1}}]$ which is the discounted present value of holding the option rather than exercising it at time t_i. We denote the continuation value as:

$$\begin{aligned} \mathcal{C}_{t_{i-1}} &= \mathbb{E}[V_{t_i}|X_{t_{i-1}}] \\ & i = 1, 2, 3, ..., N-1 \end{aligned} \tag{5}$$

Since the payoff $h(X_{t_N}) \geq 0$ is always non-negative, we can set $\mathcal{C}_{t_N} = 0$ i.e. at expiry it is no longer optimal to hold the option. Conversely, we obtain the value function V_{t_i} for $i = 1, ..., N$:

$$V_{t_i} = \max\{h(X_{t_i}), \mathcal{C}_{t_i}\} \tag{6}$$

At the last step $t_0 = 0$, we estimate the option value as:

$$\bar{V}_0 = \frac{1}{M} \sum_{j=1}^{M} V_{t_0}^{(j)} \tag{7}$$

where M is the number of simulated paths of the Heston process X_t.

2.2. Kriging Approach

The continuation value $\mathcal{C}$ remain a 'Black box' and a pivotal piece in the valuation of American option. In the Monte Carlo framework, the continuation value can be viewed as following estimation problem:

$$\begin{aligned} V_{t_i} &= \tilde{\mathcal{C}}_{t_{i-1}} + \epsilon_{t_{i-1}} \\ &= f_{i-1}(X_{t_{i-1}}) + \epsilon_{t_{i-1}} \end{aligned} \tag{8}$$

where f_{i-1} is a measurable function and $\epsilon_{t_{i-1}}$ is the error term with mean 0 and variance $\sigma_{t_{i-1}}$.

Longstaff and Schwartz [8] introduced an innovative parametric approach to derive the continuation value. For in the money paths, they estimate $\mathcal{C}$ via linear combination of K basis function $\phi_k(X)$ which are dependent on the underlying stochastic process X:

$$\tilde{\mathcal{C}}(X_{t_i}) = \sum_{k=1}^{K} \beta_{k,i-1}\phi_k(X_{t_{i-1}}) = \Phi(X_{t_{i-1}})^T\beta_{i-1} \tag{9}$$

with β_{i-1} minimizing error term $\epsilon_{t_{i-1}}$ in equation (8):

$$\beta_{i-1} = (\mathbb{E}[\Phi(X_{t_{i-1}})\Phi(X_{t_{i-1}})^T])^{-1}\mathbb{E}[\Phi(X_{t_{i-1}})V_{t_i})]. \tag{10}$$

Although LSM is a reliable and precise algorithm, finding the sufficient number of simulations and optimal number of basis function [4] is becoming more difficult to perform with higher dimensional stochastic processes. For this purpose, we introduce a non-parametric approach Gaussian process regression.

A Gaussian process (GP) is a collection of random variables where any subset has a Gaussian distribution. A Gaussian process:

$$f(X) \sim \mathcal{GP}(m(X), \mathcal{K}(X, X^T)) \tag{11}$$

is only defined by the mean $m(X)$ and covariance matrix $\mathcal{K}(X, X^T)$ for a positive and symmetric kernel $\mathcal{K}$. There are many choices for kernel but one of most preferred kernels is the squared exponential function:

$$\mathcal{K}(x, y) = \sigma_f^2 \exp\left(-\frac{1}{2l^2}||x-y||_2^2\right) \tag{12}$$

Where the hyperparameters σ_f and l are found by maximizing the log-likelihood. Since this kernel is symmetric and positive, it produces positive definite and symmetric covariance matrix.

Due to the flexibility of GPR, we don't need to use all the simulated paths to estimated all the parameters. We select a random subset $X'_{t_{i-1}}$ of the simulated paths of size M' and V'_{t_i} the corresponding path-wise option values. Using equation (8), we can represent the option value as a Gaussian distribution:

$$V'_{t_i} \sim \mathcal{N}\left(m(X'_{t_{i-1}}), \mathcal{K}(X'_{t_{i-1}}, X'^T_{t_{i-1}}) + \sigma_{t_{i-1}}I\right) \tag{13}$$

We have to find the optimal hyperparameters $\eta = (\sigma_f, l, \sigma_{t_{i-1}})$. For this reason we introduce the marginal likelihood $p(V'_{t_i}|X'_{t_{i-1}})$:

$$p(V'_{t_i}|X'_{t_{i-1}}) = \int p(V'_{t_i}|f, X'_{t_{i-1}})p(f|X'_{t_{i-1}})df \tag{14}$$

We have $V'_{t_i} = f(X'_{t_{i-1}}) + \epsilon(X'_{t_{i-1}})$ and $\epsilon(X'_{t_{i-1}}) \sim \mathcal{N}(0, \sigma_{t_{i-1}}) \Rightarrow V'_{t_i}|f(X'_{t_{i-1}}) \sim \mathcal{N}(f, \sigma_{t_{i-1}}I)$. We get prior distribution $f(X')|X' \sim \mathcal{N}(m(x), \mathcal{K}(X', X'))$. Finally, we arrive at log-likelihood:

$$\begin{aligned} \log(p(V'_{t_i}|X'_{t_{i-1}}, \eta)) = \quad & -\frac{1}{2}V'^T_{t_i}[\mathcal{K}(X'_{t_{i-1}}, X'_{t_{i-1}}) + \sigma_{t_{i-1}}I]^{-1}V'_{t_i} \\ & -\frac{1}{2}\log[\mathcal{K}(X'_{t_{i-1}}, X'_{t_{i-1}}) + \sigma_{t_{i-1}}I] - \frac{M'}{2}\log(2\pi) \end{aligned} \tag{15}$$

By maximizing the log-likelihood in (15), we obtain the desire parameters $\eta = (\sigma_f, l, \sigma_{t_{i-1}})$ for the Gaussian process regression. We have divided the total number of simulations into two sets $X'_{t_{i-1}}$ (learning set) and the complement $X^*_{t_{i-1}}$ (prediction set). If we apply same estimates to the prediction set we get a noise-free estimation of the continuation value for the entire set:

$$\begin{bmatrix} f(X'_{t_{i-1}}) \\ f(X^*_{t_{i-1}}) \end{bmatrix} \sim \mathcal{N}\left(\begin{matrix} m(X'_{t_{i-1}}) \\ m(X^*_{t_{i-1}}) \end{matrix}, \begin{bmatrix} \mathcal{K}(X'_{t_{i-1}}, X'^T_{t_{i-1}}) & \mathcal{K}(X'_{t_{i-1}}, X^{*T}_{t_{i-1}}) \\ \mathcal{K}(X^*_{t_{i-1}}, X'^T_{t_{i-1}}) & \mathcal{K}(X^*_{t_{i-1}}, X^{*T}_{t_{i-1}}) \end{bmatrix} \right)$$

A more detailed derivation on GPR is discussed by Rasmussen and Williams [14].

Employing the GPR method, we obtain an estimation of continuation value:

$$\tilde{\mathcal{C}}_{t_{i-1}} = f(X_{t_{i-1}}) = \mathcal{GP}_t(m(X_{t_{i-1}}), \mathcal{K}(X_{t_{i-1}}, X^T_{t_{i-1}})) \tag{16}$$

We can write $\tilde{\mathcal{C}}_{t_{i-1}} = \mathcal{P}_{t_{i-1}} V_{t_i}$ where $\mathcal{P}_{t_{i-1}}$ is the projection operator onto $\mathcal{F}_{t_{i-1}}$-measurable Gaussian Processes. We can also apply the learning stage on a smaller set than total amount of paths generated because GPR is computationally heavy with $O(M^3)$ operations.

Lastly, we improve the Kriging approach by including control variates (CV) in similar fashion as Rasmussen N. S. [13]. We use European put options control variates $Y_T = h(X_T)$ with maturity time $T = t_N$ written on the same underlying stochastic process and apply it directly to continuation value. In terms Gaussian process regression, we adjust the continuation value with the projection operator $\mathcal{P}_t$, in the following way:

$$\mathcal{C}^{CV}_{t_{i-1}} = \mathcal{P}_{t_{i-1}} V_{t_i} - \tilde{\nu}_{t_{i-1}} (\mathcal{P}_{t-1} Y_{t_i} - \mathbb{E}[h(X_{t_N}) | X_{t_{i-1}}]) \tag{17}$$

where $\tilde{\nu}_{t_{i-1}}$ minimizes the variance of $\mathcal{C}^{CV}_{t_{i-1}}$:

$$\tilde{\nu}_{t_{i-1}} = \frac{\mathcal{P}_{t_{i-1}}(V_{t_i} Y_{t_i}) - \mathcal{P}_{t_{i-1}} V_{t_i} \cdot \mathcal{P}_{t_{i-1}} Y_{t_i}}{\mathcal{P}_{t_{i-1}}(Y^2_{t_i}) - (\mathcal{P}_{t_{i-1}} Y_{t_i})^2} \tag{18}$$

We calculate the discounted price of European option $p_{t_{i-1}} = \mathbb{E}[h(X_{t_N}) | X_{t_{i-1}}]$ at each path via fast Fourier transform method described by Carr and Madan [2]. At each step, we adjust the control variate as $Y_{t_N} = h(X_{t_N})$ $Y_{t_{i-1}} = p_{t_{i-1}}$ if $h(X_{t_{i-1}}) > \mathcal{C}^{CV}_{t_{i-1}}$ otherwise $Y_{t_{i-1}} = Y_{t_i}$. At the last step, the control variate Y_{t_0} is applied directly to the value of the option V_{t_0}.

3. Numerical Study

We design the numerical test to compare the three mentioned algorithms: LSM, Kriging and Kriging with CV over a set of different parameters. In the first part, we use standard examples and benchmarks from literature [12] to test the algorithms' efficiency. In the second part, we compare the methods on realistic parameters by first calibrating the Heston model to the market data from CBOE on Russell 2000 Index Options (RUT) and S&P 500 Index Options (SPX).

3.1. Synthetic Data Experiment

Let $t_0 = 0, t_1, ..., t_N = T$ be the equidistant discretization of the interval $[0, T]$ with the difference $\Delta t = \frac{T}{N}$. Then for $i = 1, 2, ..., N$, we can rewrite the discrete risk-neutral Heston dynamics (1) for sufficiently small Δt:

$$\begin{aligned} \log S_{t_i} &= \log S_{t_{i-1}} + (r - \frac{1}{2} v_{t_{i-1}})\Delta t + \sqrt{v_{t_{i-1}}}\sqrt{\Delta t}\epsilon_{S,t_i} \\ v_{t_i} &= v_{t_{i-1}} + \kappa^*(\theta^* - v_{t_{i-1}})\Delta t + \sigma\sqrt{v_{t_{i-1}}}\sqrt{\Delta t}(\rho\epsilon_{S,t_i} + \sqrt{1-\rho^2}\epsilon_{v,t_i}) \end{aligned} \tag{19}$$

where $\{\epsilon_{S,t_i}, \epsilon_{v,t_i}\}_{i=1,...,N}$ are i.i.d. random variables with standard normal distribution. This discretization method is also known as Euler–Maruyama. We replicate the correlation $< dW_{1,t}^Q, dW_{2,t}^Q >= \rho$ by constructing the random variable $< \epsilon_{S,t}, \rho\epsilon_{S,t} + \sqrt{1-\rho^2}\epsilon_{v,t} >= \rho$ which has the same correlation and behavior. We replace the terms $v_{t_{i-1}}$ with $max(v_{t_{i-1}}, 0)$ due to the fact that v_{t_i} in discrete approximation may reach negative values for some extreme states of volatility.

For the three methods, we use backward dynamic programming formulation in (4) but we update the option value function:

$$V_{t_{i-1}} = \begin{cases} h(S_{t_{i-1}}) & \text{if } h(S_{t_{i-1}}) \geq C_{t_{i-1}} \\ V_{t_i} & \text{otherwise} \end{cases} \tag{20}$$

In this way, we don't add the errors from the estimation of continuation value in the option value function calculations.

In the LSM algorithm, we estimate the continuation value $\mathcal{C}$ as in equation (9) with the following basis functions in terms of stock prices S and volatility v:

$$\Phi(S, v) = \{1, S, v, Sv, S^2, v^2, Sv, S^3, v^3, S^2v, Sv^2\}. \tag{21}$$

The interaction terms between S and v are necessary to capture the correlation ρ of the Brownian motions of the stock price and its volatility. For the Kriging algorithm, we are estimating the continuation value $\mathcal{C}$ via Gaussian process regression (16) on the basis function $\{S, v\}$ and a random learning subset of paths of size M_0. Whereas, in the Kriging with CV we adjusting previous continuation value $\mathcal{C}$ as in equation (17), performing the other 3 GPR on the same set of basis functions.

In Table 1, we present the results for $S_0 \in \{90, 100, 110\}$, $T \in \{0.25, 0.5\}$ and $v_0 \in \{0.04, 0.09, 0.16\}$ to account for in the money (ITM), at the money (ATM) and out of the money options (OTM) options with different maturities T and riskiness v_0. For all of the test, GPR algorithm approximates the American options prices with a precision of up to 3 fold than LSM. With the addition of control variate, GPR algorithm further improves the precision (up to 10 fold) especially for high volatility, OTM options and high maturity. In Figure 1, we show that all 3 algorithm converge slowly as the number of simulated paths M increases with $O(\sqrt{M})$ speed. The GPR-CV algorithm reaches a suitable error even with M as low as 2000. This showcases that Kriging with CV algorithm needs little information to output high precision results.

S_0	v_0	T	PSOR	LSM Price	LSM Error %	LSM Std	GPR Price	GPR Error %	GPR Std	GPR-CV Price	GPR-CV Error %	GPR-CV Std
90	0.04	0.25	10.1229	10.1653	0.4181	4.5720	10.1513	0.2806	4.3806	10.0990	0.2362	0.2808
100	0.04	0.25	3.4813	3.5240	1.2258	4.1948	3.5068	0.7329	4.2624	3.4632	0.5212	0.1663
110	0.04	0.25	0.8417	0.8741	3.8418	2.4929	0.8735	3.7729	2.4695	0.8282	1.6069	0.0929
90	0.09	0.25	10.9573	11.0423	0.7756	7.0613	11.0121	0.4997	6.8796	10.9374	0.1818	0.2927
100	0.09	0.25	4.9461	5.0244	1.5824	6.3102	4.9920	0.9284	6.0819	4.9339	0.2479	0.1751
110	0.09	0.25	1.8641	1.9244	3.2331	4.2084	1.9068	2.2883	4.1085	1.8581	0.3249	0.0918
90	0.16	0.25	12.1200	12.2376	0.9703	9.1855	12.1845	0.5320	8.8658	12.1059	0.1163	0.2680
100	0.16	0.25	6.4933	6.6016	1.6676	8.0982	6.5553	0.9551	7.8332	6.4841	0.1409	0.1753
110	0.16	0.25	3.1470	3.2385	2.9067	6.0491	3.2003	1.6923	5.8564	3.1444	0.0820	0.1061
90	0.04	0.5	10.5667	10.6314	0.6127	5.9602	10.6110	0.4198	5.8109	10.5373	0.2782	0.5463
100	0.04	0.5	4.6645	4.7266	1.3311	5.9187	4.6846	0.4295	5.6213	4.6460	0.3963	0.3512
110	0.04	0.5	1.7875	1.8421	3.0558	4.0904	1.8262	2.1650	3.9732	1.7741	0.7504	0.1807
90	0.09	0.5	11.7658	11.8552	0.7595	8.3687	11.8166	0.4318	8.0841	11.7463	0.1654	0.5730
100	0.09	0.5	6.2573	6.3427	1.3652	7.6451	6.2935	0.5799	7.3566	6.2435	0.2195	0.4012
110	0.09	0.5	3.0673	3.1497	2.6870	5.8725	3.1049	1.2273	5.6399	3.0602	0.2318	0.2432
90	0.16	0.5	13.2329	13.3473	0.8642	10.4700	13.2823	0.3733	10.0642	13.2159	0.1285	0.5669
100	0.16	0.5	8.0073	8.1207	1.4158	9.5221	8.0532	0.5739	9.1627	7.9973	0.1243	0.4255
110	0.16	0.5	4.6232	4.7312	2.3360	7.8025	4.6693	0.9967	7.4842	4.6194	0.0829	0.2936

Table 1. Convergence of American put option prices computed by Least Square Method, Gaussian process regression with/without control variate averaged over 100 independent runs versus PSOR Benchmark. Parameters: $K = 100$, $r = 0.05$, $\kappa = 3$, $\theta = 0.04$, $\sigma = 0.1$, $\rho = -0.7$, $N = 100$,$M = 5000$ and $M_0 = 2000$

3.2. Market Data Experiment

We aim to price American options on RUT and SPX indexes with LSM and GPR-CV. We select European put and call option prices of the 2 indexes at "24-08-2015" having stock prices $S_{RUT} = 1111.69$ and $S_{SPX} = 1893.2$, with maturity "19-12-2015" (T=0.32) and "18-03-2016" (T=0.57) and with strike prices K in range of 10% of the stock price.

For both indices, we derive the dividend yield q and interest rate r using Put-Call parity on the market data prices via a linear regression as in Ackerer [1]:

$$C_{mkt} - P_{mkt} = e^{-qT} S_0 - e^{-rT} K. \tag{22}$$

where C_{mkt}, P_{mkt} are the market prices of the call and put with strike price K and maturity T on index S_0. We calibrate the parameters of the Heston model by minimizing the mean squared distance between the Black-Scholes implied volatility of market option prices and Heston option prices [1]:

$$\min_{\{\sigma, v_0, \kappa, \theta, \rho\}} \sqrt{\frac{1}{|\mathbb{O}|} \sum_{o \in \mathbb{O}} (\sigma_{mkt}^{imp} - \sigma_{Heston}^{imp})^2} \tag{23}$$

where $\mathbb{O}$ are the set of options on the same index, σ_{mkt}^{imp} is the implied volatility of market price of the option and σ_{Heston}^{imp} is the implied volatility of the option price

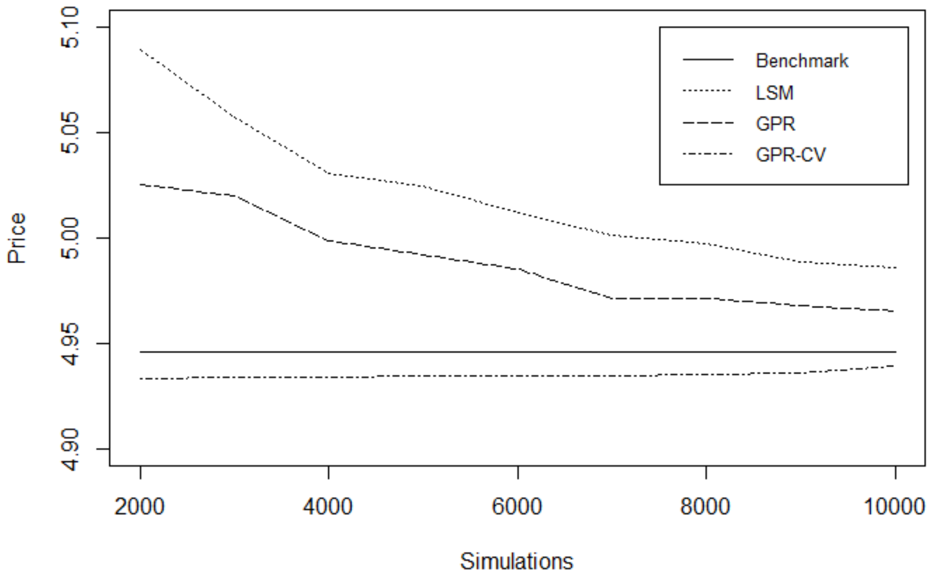

Figure 1. American put option price behavior averaged over 100 independent runs. Parameters: $S_0 = 100$, $K = 100$, $T = 0.25$, $v_0 = 0.09$, $r = 0.05$, $\kappa = 3$, $\theta = 0.04$, $\sigma = 0.1$, $\rho = -0.7$, $N = 100$ and $M_0 = 2000$

calculated via Car-Madan formula on Heston model. As a result, we obtain the following parameters:

Index	r	q	$r-q$	σ	v_0	κ	θ	ρ
RUT	0.0229	0.0057	0.0171	0.8669	0.1098	6.4541	0.0487	-0.5751
SPX	0.0211	0.0060	0.0151	0.8497	0.1070	6.6356	0.0302	-0.5630

Table 2. Calibrated Parameters

With the calibrated parameters, we apply the same procedure as in Section 3.1 to price the American options on RUT and SPX indices and report the results in the Table 3. We arrive at the same conclusion that GPR-CV algorithm outperforms LSM algorithm especially for OTM options.

4. Conclusions

We have laid down the backward dynamic principle for American options pricing in the context of Monte Carlo methods. We have presented the LSM and Kriging approach in

Index	K	T	PSOR*	LSM Price	LSM Error %	LSM Std	GPR-CV Price	GPR-CV Error %	GPR-CV Std
RUT	1000	0.32	25.396	26.443	4.1219	62.190	25.411	0.0606	1.111
	1110	0.32	61.684	63.057	2.2260	91.084	60.892	1.2830	2.989
	1220	0.32	127.339	129.190	1.4535	110.985	125.774	1.2292	3.624
	1000	0.57	36.524	37.902	3.7715	78.755	36.484	0.1114	1.862
	1110	0.57	75.654	77.315	2.1956	107.009	74.502	1.5230	4.028
	1220	0.57	138.775	140.580	1.3003	126.566	136.353	1.0245	5.352
SPX	1700	0.32	35.983	37.668	4.6825	95.029	36.013	0.0845	1.868
	1900	0.32	99.744	102.115	2.3763	146.753	98.129	1.6194	5.538
	2100	0.32	226.829	229.756	1.2905	176.642	224.018	1.2390	6.578
	1700	0.57	49.137	51.341	4.4850	117.266	49.105	0.0651	3.164
	1900	0.57	116.784	119.359	2.2041	166.954	114.949	1.5714	7.759
	2100	0.57	237.415	241.384	1.6720	196.110	234.265	1.3269	9.638

Table 3. Performance of LSM, GPR-CV on RUT and SPX versus PSOR benchmark. Parameters from Table 2 and $N = 100$,$M = 5000$, $M_0 = 2000$.
* *American Option price calculated via explicit scheme with discretization* $\Delta_S = 2^{10}$, $\Delta_v = 2^9$ *and* $\Delta_T = 2^{20}$.

this framework together with control variate adjustment. In particular, we have shown that GPR method always outperforms LSM algorithm especially when combined with European counterpart as a control variate but at an additional computational cost. The numerical results on OTM options illustrate that GPR-CV is well equipped to deal with minimal amount of information and provide precise pricing results.

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