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PREDICTING FATIGUE CRACK GROWTH AND FATIGUE LIFE UNDER VARIABLE AMPLITUDE LOADING

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Abstract

A probabilistic approach to the description of fatigue crack growth and fatigue life estimation of a component subjected to variable amplitude loading is presented in the paper. The core of the model is a differential equation originated from the Paris formula. In order to consider the influence of overload-underload cycles existing in an exploitive load spectrum on crack growth rate for an aeronautical aluminum alloy sheet, the modified Willenborg retardation model was applied.

1. INTRODUCTION

The operational spectrum of a structure is a typical variable amplitude spectrum. Exploitive loading induces in the materials physical phenomena that influence crack growth behavior. This is known as the effect of load interaction, which means the importance of both initial crack length at a given moment and the load time history for the crack growth in the materials. There exist numerous physical mechanisms that accompany the crack extension under single or multiple overloads and underloads imposed cyclically or randomly in the base line load. The most frequently mentioned mechanisms are either plastically induced crack closure and crack rate retardation associated with the plastic zone ahead of a crack tip that it was induced by a tensile overload cycle. Compressive underload cycle, on the other hand, leads to the crack tip sharpening and the crack rate increasing. The distribution of residual stresses in the plastic zone, the thickness of a particular component as well as mechanical properties of the material are determining factors that contribute to irregular fatigue crack growth. Therefore, it is of considerable interest to quantitatively predict the experimental tendency in crack growth behavior due to changes in load, material and geometry of a component. For this goal, certain empirical prediction models (Elber, Wheeler, Willenborg) as well as numerical simulations of crack growth under variable amplitude load derived by the codes FASTRAN, NASGRO, CORPUS and AFGROW find application. The calculative model for predicting crack growth rate in the 2024-T3 aluminum alloy under block program loading of the low-high-low type can be found in [1].

The influence of the shape of the loading spectrum on the crack rate is analyzed by means of the electron microscopes SEM and TEM. A local crack growth rate is estimated on the basis of striation spacing measurement. The details of the microfractographic analysis concerning the 2024-T3 alloy and its correlation with the fatigue crack growth rate under single and multiple overloads-underloads can be found in [2].

In the present paper, a probabilistic approach to predicting fatigue crack growth rate under variable amplitude loading with imposing multiple overload-underload cycles was developed on the basis of the modified Willenborg model.

2. PROBABILISTIC METHOD OF FATIGUE LIFE ESTIMATION

For predicting fatigue crack growth rate in a component subjected to random loading, a probabilistic model is proposed. The reasons for applying the probabilistic approach are as follows:

- inhomogeneity of the real material,
- scatter of mechanical properties of the material,
- randomness of cracking process,
- technological conditions (quality of manufacturing).

There, it is required that the model and the real object are physically identical as regards the time and point of crack initiation, crack propagation period and fatigue lifetime of a component. Generally, exploitive loading contains a wide spectrum of stress cycles such as base line cycles, overloads and underloads, which appear in different order. The application of single or multiple tensile overloads causes significant decrease in the crack growth rate for a large number of cycles subsequent to the value of overload. It results from the compressive residual stresses acting in the plastic zone ahead of the crack tip. Application of compressive underloads has a detrimental effect on crack initiation and crack growth. The crack growth rate shows increasing trend and fatigue life will be reduced.

In order to calculate the retardation effect on the crack rate due to overload-underload cycles, the improved Wheeler model known as the Willenborg model was applied. Both these models are based on the assumption that crack growth is controlled not only by the plastic zone but also by residual deformation left in the wake of the crack as it grows through the previously deformed material [3]. In the Willenborg model, there was introduced a reduced stress σ_{red}, which is needed to get through the plastic zone $r_{p,ol}$ created by the tensile-overload cycle. In the case of an underload half-cycle with compressive stress, either the current plastic zone of the radius $r_{p,i}$ or the overload plastic zone of the radius $r_{p,ol}$ ahead of the crack tip are reduced by the radius r_{cp} . In accordance with [3], the plastic zone radius r_{cp} , which results from the interaction of an elastic material in the vicinity of growing crack, is determined by the equation (1):

$$r_{cp} = \frac{1}{D\cdot\pi}\left(\frac{\Delta K_{UL}}{2\cdot R_{02}}\right)^2 = \frac{1}{D\cdot\pi}\left(\frac{K^* - K_{\min,UL}}{2\cdot R_{02}}\right)^2 \tag{1}$$

Where: D = 2 (plane stress state) or 6 (plane strain state), $K^* = \min(K_{\min}^{CA}, K_{th})$; $K_{\min}^{CA}$ is the minimum stress intensity factor in a base CA cycle; K_{th} is the threshold stress intensity factor; $K_{min,UL}$ is the minimum K associated with the underload (UL); $R_{0,2}$ is the yield stress.

The crack growth retardation effect continues as long as the following relation is true:

$$a_i + (r_{p,i} - r_{cp,i}) < a_{ol} + (r_{p,OL} - r_{cp,UL}) \tag{2}$$

strictly speaking, the crack growth is delayed as long as the current crack of the length a_i enlarged by the current monotonic plastic zone of the radius ($r_{p,i}$ - $r_{cp,i}$) does not go beyond the overload crack of the length a_{OL} enlarged by the plastic zone of the radius ($r_{p,OL}$ - $r_{cp,UL}$). Therefore, the retardation factor Cp is given by the equation:

$$C_p = \left(\frac{r_{pi} - r_{cpi}}{a_{OL} + (r_{p,OL} - r_{cp,UL}) - a_1}\right)^n \tag{3}$$

The stress redistribution occurs ahead of the crack tip as a result of a tensile reaction of an elastic material surrounding the growing crack and the compressive stresses acting in the monotonic plastic zone ahead of this crack. Assuming that the reduced stresses σ_{red} operate in the plastic zone, the condition for the crack growth retardation in the Willenborg model is as follows:

$$a_{OL} + (r_{po} - r_{cp,UL}) = a_i + \frac{1}{D \cdot \pi}\left(\frac{K_{red}}{R_{0,2}}\right)^2 = a_i + \frac{1}{D \cdot \pi}\left(\frac{\sigma_{red} \cdot \sqrt{\pi \cdot a_i \cdot M_k}}{R_{0,2}}\right)^2 \tag{4}$$

The stress required for getting through the plastic zone is determined by the equation:

$$\sigma_{red} = \frac{\sqrt{2} \cdot R_{0,2}}{M_k} \cdot \sqrt{\frac{a_{ol} + \left(r_{OL} - r_{cp,UL}\right) - a_i}{a_i}} \tag{5}$$

where M_k is the geometrical factor.

In the model, it is assumed that the value of compressive stresses σ_c existing in the overload plastic zone is equal to the reduced stress σ_{red} minus the maximum applied overload stress, that is $\sigma_c = \sigma_{red} - \sigma_{max,OL}$. The values of $\sigma_{max,i}$ and $\sigma_{min,i}$ are reduced by the compressive stress σ_c in each load cycle. In a fatigue cycle, the values of effective maximum $\sigma_{max\ eff,j}$ and minimum stresses $\sigma_{min\ eff,j}$ equal respectively to:

$$\begin{aligned}\sigma_{\max eff,j} &= \sigma_{\max,j} - \sigma_c = 2\sigma_{\max,j} - \sigma_{red} \\ \sigma_{\min eff,j} &= \sigma_{\min,j} - \sigma_c = \sigma_{\min,j} + \sigma_{\max,j} - \sigma_{red}\end{aligned} \tag{6}$$

The values of the above effective stresses must be positive numbers, otherwise it should be assumed that the value is zero.

The effect of the stress ratio R on crack growth rate should be taken into account while calculating effective stress changes according to the equation:

$$\Delta\sigma_{eff,j} = \sigma_{\max eff,j} \cdot \left(1 - R_j\right)^{\gamma} \quad j = 1 \ldots q, \quad R_j \geq 0 \tag{7}$$

where $\sigma_{max\ eff,j}$ and R_j are the maximum effective stress and the stress ratio for j-th stress block, respectively. The value of the stress modification factor γ is 0.68 for aluminum alloys under variable-amplitude loading with the stress ratio $R \geq 0$.

Let's assume that the crack growth rate follows by the Paris formula under each stress cycle:

$$\Delta K_j = M_{k,j} \cdot \Delta\sigma_{eff,j} \cdot \sqrt{\pi \cdot a_j} \tag{8}$$

where ΔK_j is the stress intensity factor for the length a_j of the crack.

Further assumptions are as follows:

1. load spectrum consists of N_c cycles,

   $$N_c = \sum_{j=1}^{L} n_j$$

   where n_j denotes the number of cycles having a given stress range $\Delta\sigma j$,
2. load cycles can be ordered in L stress levels $\Delta\sigma j$, j =1,2,…,L. Each load level has the same maximum and minimum stress,
3. the values of both stress ranges and the frequency of stress level appearing in the spectrum are performed in the table 1.

Tab. 1.

stress range	$\Delta\sigma_1$	$\Delta\sigma_2$	...	$\Delta\sigma_L$
frequency of occurrence of a given stress level in the load spectrum	$\frac{n_1}{N_c} = P_1$	$\frac{n_2}{N_c} = P_2$	...	$\frac{n_L}{N_c} = P_L$

where $P_1 + P_2 + \ldots + P_L = 1$, and P_j are the probability of a given stress cycle occurrence in a load spectrum

4. let's assume that the crack advances according to the Paris formula:

$$\frac{da}{dN} = U \cdot C \cdot (\Delta K_j)^m \tag{9}$$

where ΔK_j is the range of stress intensity factor determined by equation (8), C, m are material constants, a is crack length; N - number of load cycles.

Empirical function U of crack closure contribution to crack growth relates to stress ratio R $U = 0,55 + 0,33R + 0,12R^2$.

Taking into account equations (8) and (9) then Paris formula can be expressed in the form:

$$\frac{da}{dN} = UC M_K^m (\Delta\sigma_{eff,j})^m \pi^{\frac{m}{2}} a^{\frac{m}{2}} \tag{10}$$

where: M_k is geometrical coefficient and the stress $\Delta\sigma_{eff,j}$ is defined by relation (7).

Substituting the relation $N=\lambda\ t$ to equation (10) we arrive to the Paris formula expressed as function of time:

$$\frac{da}{dt} = \lambda \cdot U \cdot C \cdot M_K^m \cdot (\Delta\sigma_{eff,j})^m \cdot \pi^{\frac{m}{2}} \cdot a^{\frac{m}{2}} \tag{11}$$

where λ means the frequency of load cycle appearing in a spectrum and t is a time of the loading action.

In order to clearly perform the probabilistic method of estimating both the fatigue crack growth and fatigue life of a component under variable amplitude loading, four main steps will be distinguished.

Step 1
Dynamics of crack growth expressed by difference equation

Stochastic nature of crack growth is expressed by a function $U_{a,t}$ representing the probability that at the time t the crack length is a [4]. For this, the dynamics of crack growth was described by difference equation:

$$U_{a,\,t+\Delta t} = P_1 U_{a-\Delta a_1,\,t} + P_2 U_{a-\Delta a_2,\,t} + \ldots + P_L U_{a-\Delta a_L,\,t} \tag{12}$$

where the crack length increment Δa results from the action of stress $\Delta\sigma_{eff,\,j}$ occurring with the probability P_j.

Difference equation (12) in functional notation took the following form:

$$u(a,\, t+\Delta t) = \sum_{i=1}^{L} P_i u\ (a - \Delta a_i,\, t) \tag{13}$$

where $u(a,t)$ means the crack length density function depending on time t.

Step 2
Transformation of difference equation into the Fokker-Planck differential equation

Finite difference equation (13) was transformed into differential equation by expanding it into Taylor series :

$$\left.\begin{aligned} u(a,\, t+\Delta t) &= u(a,t) + \frac{\partial u(a,t)}{\partial t}\Delta t; \\ u(a-\Delta a_i,\, t) &= u(a,\, t) - \Delta a_i \frac{\partial u(a,\, t)}{\partial a} + \frac{1}{2}(\Delta a_i)^2 \frac{\partial^2 u(a,t)}{\partial a^2} \end{aligned}\right. \quad \text{for } i = 1, 2, \ldots, L \tag{14}$$

Equations (14) after transformation have following form:

$$\frac{\partial u(a,t)}{\partial t} = -\lambda(P_1\Delta a_1 + P_2\Delta a_2 + \ldots + P_L\Delta a_L)\frac{\partial u(a,t)}{\partial a} + \\ + \frac{1}{2}\lambda(P_1(\Delta a_1)^2 + P_2(\Delta a_2)^2 + \ldots + P_L(\Delta a_L)^2)\ \frac{\partial^2 u(a,t)}{\partial a^2} \tag{15}$$

Transformation to functional notation and the Taylor series expansion deliver the Fokker-Planck type equation (16)

$$\frac{\partial u(a,t)}{\partial t} = -\alpha(a)\frac{\partial u(a,t)}{\partial a} + \frac{1}{2}\beta(a)\frac{\partial^2 u(a,t)}{\partial a^2} \tag{16}$$

where $u(a,t)$ is the crack length density function depending on time.

$$\begin{aligned} \alpha(a) &= \lambda\sum_{i=1}^{L} P_i\Delta a_i, \\ \beta(a) &= \lambda\sum_{i=1}^{L} P_i(\Delta a_i)^2, \\ \Delta a_i &= C_m(\Delta\sigma_{eff,i})^m a^{\frac{m}{2}}, \\ C_m &= UC\,M_K^m\,\pi^{\frac{m}{2}} \end{aligned} \tag{17}$$

For $m \neq 2$ coefficient $\alpha(a)$ in equation (16) has following form:

$$\begin{aligned} \alpha(a) &= \lambda[P_1C_m(\Delta\sigma_{eff,1})^m a^{\frac{m}{2}} + P_2C_m(\Delta\sigma_{eff,2})^m a^{\frac{m}{2}} + \ldots + P_LC_m(\Delta\sigma_{eff,L})^m a^{\frac{m}{2}}] = \\ &= \lambda C_m\, a^{\frac{m}{2}} \underbrace{[P_1(\Delta\sigma_{eff,1})^m + P_2(\Delta\sigma_{eff,2})^m + \ldots + P_L(\Delta\sigma_{eff,L})^m]}_{E[(\Delta\sigma_{eff})^m]} = \\ &= \lambda C_m\, E[(\Delta\sigma_{eff})^m]a^{\frac{m}{2}} = \lambda UCM_K^m\pi^{\frac{m}{2}}E[(\Delta\sigma_{eff})^m]\cdot a^{\frac{m}{2}} \end{aligned} \tag{18}$$

where:

$$E[(\Delta\sigma_{eff})^m] = P_1(\Delta\sigma_{eff,1})^m + P_2(\Delta\sigma_{eff,2})^m + \ldots + P_L(\Delta\sigma_{eff,L})^m \tag{19}$$

Coefficient $\beta(a)$ in equation (16) has following form:

$$\begin{aligned} \beta(a) &= \lambda\sum_{i=1}^{L} P_i(\Delta a_i)^2 \\ \beta(a) &= \lambda\{[P_1(C_m(\Delta\sigma_{eff,1})^m a^{\frac{m}{2}})^2 + P_2(C_m(\Delta\sigma_{eff,2})^m a^{\frac{m}{2}})^2 + \ldots + P_L(C_m(\Delta\sigma_{eff,L})^m a^{\frac{m}{2}})^2]\} = \\ &= \lambda[P_1C_m^2(\Delta\sigma_{eff,1})^{2m} a^m + P_2C_m^2(\Delta\sigma_{eff,2})^{2m}\cdot a^m + \ldots + P_LC_m^2(\Delta\sigma_{eff,L})^{2m}\cdot a^m)^2] = \\ &\underbrace{[P_1(\Delta\sigma_{eff,1})^{2m} + P_2(\Delta\sigma_{eff,2})^{2m} + \ldots + P_L(\Delta\sigma_{eff,L})^{2m}]}_{E\left[(\Delta\sigma_{eff})^{2m}\right]}\lambda\, C_m^2\cdot a^m = \\ &= \lambda C_m^2\, E[(\Delta\sigma_{eff})^{2m}]a^m = \lambda\, U^2C^2M_K^{2m}\pi^m E[(\sigma_{eff})^{2m}]a^m \end{aligned} \tag{20}$$

Integrating of equation (11) allows us to deliver the crack length a which is presented in equations (18) and (20) for the assumption $m \neq 2$:

$$\begin{aligned}
&\frac{da}{dt} = \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \cdot \pi^{\frac{m}{2}} \cdot a^{\frac{m}{2}} \\
&\frac{1}{a^{\frac{m}{2}}} da = \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \cdot \pi^{\frac{m}{2}} \cdot dt \\
&\int_{a_o}^{a} a^{-\frac{m}{2}} da = \int_0^t \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} \cdot dt \\
&\frac{2}{2-m} a^{\frac{2-m}{2}} \Bigg|_{a_0}^{a} = \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} \cdot t \\
&\frac{2}{2-m} (a^{\frac{2-m}{2}} - a_o^{\frac{2-m}{2}}) = \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} \cdot t \\
&a^{\frac{2-m}{2}} - a_o^{\frac{2-m}{2}} = \frac{2-m}{2} \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \cdot \pi^{\frac{m}{2}} \cdot t \\
&a = (a_o^{\frac{2-m}{2}} + \frac{2-m}{2} \lambda \cdot U \cdot C \cdot M_K^m \cdot E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} t)^{\frac{2}{2-m}}
\end{aligned} \tag{21}$$

Equation (21) applied to equation (18) allowed determination of the coefficient $\alpha(a)$ dependent on time t:

$$\alpha(t) = \lambda U C M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \cdot [(a_o^{\frac{2-m}{2}} + \frac{2-m}{2} \lambda U C M_K^m E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} t)]^{\frac{m}{2-m}} \tag{22}$$

Equation (21) applied to equation (20) allowed obtaining the form for the coefficient $\beta(a)$ dependent on time t:

$$\beta(t) = \lambda U^2 C^2 M_K^{2m} \pi^m E[(\Delta\sigma_{eff})^{2m}] \cdot \{[a_o^{\frac{2-m}{2}} + \frac{2-m}{2} \lambda U C M_K^m E[(\Delta\sigma_{eff})^m] \pi^{\frac{m}{2}} t]^{\frac{2}{2-m}}\}]^m \tag{23}$$

Step 3
Crack length density function as a solution of the Fokker-Planck type differential equation

Taking equations (22) and (23) into account, then equation (16) takes the form of:

$$\frac{u(a,t)}{\partial t} = -\alpha(t) \frac{\partial u(a,t)}{\partial a} + \frac{1}{2} \beta(t) \frac{\partial^2 u(a,t)}{\partial a^2} \tag{24}$$

We are looking for a special solution of equation (24) which satisfies the following conditions: if $t \to 0$ the solution is convergent with the Dirac function, $u(a,t) \to 0$ for $a \neq 0$ and $u(a,t) \to \infty$ in such a way that the integral of function u is equal one for $t > 0$.
Solution of equation (24) is the requested crack-length density function dependent on time:

$$u(a,t) = \frac{1}{\sqrt{2\pi A(t)}} e^{-\frac{(a-B(t))^2}{2A(t)}} \tag{25}$$

where:

- $B(t)$ is an average value of crack length for the time t,
- $A(t)$ is a variance of crack length for the time t.

Computational formulae take the following forms:

$$B(t) = \int_0^t \alpha(z)dz \tag{26}$$

$$A(t) = \int_0^t \beta(z)dz \tag{27}$$

Calculating the integral (26) one can receive:

$$B(t) = \int_0^t \alpha(z)dz =$$

$$= \lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \int_0^t (a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m E[(\Delta\sigma_{eff})^m]\pi^{\frac{m}{2}} z)^{\frac{m}{2-m}} dz =$$

$$= \lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \cdot \frac{1}{(\frac{m}{2-m}+1)} [(a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]z]^{(\frac{m}{2-m}+1)} \cdot$$

$$\cdot \frac{1}{\frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]} \Bigg|_0^t =$$

$$= \lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \cdot \frac{1}{(\frac{m}{2-m}+1)} [(a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]z]^{(\frac{m}{2-m}+1)} \cdot$$

$$\cdot \frac{1}{\frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]} \Bigg|_0^t =$$

$$= \lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \cdot \frac{1}{\left(\frac{2}{2-m}\right)} \cdot [(a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t]^{\frac{2}{2-m}} \cdot$$

$$\cdot \frac{1}{\frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]} - \lambda U C M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m] \frac{1}{(\frac{2}{2-m})} [a_o^{\frac{2-m}{2}}]^{\frac{2}{2-m}} \cdot$$

$$\cdot \frac{1}{\frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]} = [a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t]^{\frac{2}{2-m}} - [a_o^{\frac{2-m}{2}}]^{\frac{2}{2-m}}$$

Hence, $B(t)$ - an average value of crack length for the time t has the expression:

$$B(t) = [a_o^{\frac{2-m}{2}} + \frac{2-m}{2}\lambda UC M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t]^{\frac{2}{2-m}} - a_o \tag{28}$$

Calculating the integral (27) one can obtain:

$$A(t)=\int_0^t \beta(z)dz=$$

$$=\lambda U^2 C^2 M_K^{2m}\pi^m E[(\Delta\sigma_{eff})^{2m}]\cdot\int_0^t [(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda U\, C\, M_K^m E[(\Delta\sigma_{eff})^m]\pi^{\frac{m}{2}} z)^{\frac{2}{2-m}}]^m dz=$$

$$=\lambda U^2 C^2 M_K^{2m}\pi^m E[(\Delta\sigma_{eff})^{2m}]\cdot\int_0^t (a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m E[(\Delta\sigma_{eff})^m]z)^{\frac{2m}{2-m}} dz=$$

$$=\lambda U^2 C^2 M_K^{2m}\pi^m E[(\Delta\sigma_{eff})^{2m}]\left(\frac{1}{\frac{2m}{2-m}+1}\right)(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]z)^{\frac{2m}{2-m}+1}\cdot$$

$$\cdot\frac{1}{\frac{2-m}{2}\lambda C\, M_K^m\,\pi^{\frac{m}{2}} E[(\sigma^{\max})^m]}\Bigg|_0^t=$$

$$=\lambda U^2 C^2 M_K^{2m}\,\pi^m E[(\Delta\sigma_{eff})^{2m}]\,(\frac{2-m}{2+m})\,(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t]^{\frac{2+m}{2-m}}\cdot$$

$$\cdot\frac{1}{\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]}-$$

$$-\lambda U^2 C^2 M_K^{2m}\,\pi^m E[(\Delta\sigma_{eff})^{2m}]\,(\frac{2-m}{2+m})\cdot(a_o^{\frac{2-m}{2}})^{\frac{2+m}{2-m}}\frac{1}{\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]}=$$

$$=\frac{2}{2-m}CUM_K^m\,\pi^{\frac{m}{2}}\frac{E[(\Delta\sigma_{eff})^{2m}]}{E[(\Delta\sigma_{eff})^m]}\frac{(2-m)}{(2+m)}(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t)^{\frac{2+m}{2-m}}-$$

$$-\,CU\,M_K^m\,\pi^{\frac{m}{2}}\frac{E[(\Delta\sigma_{eff})^{2m}]}{E[(\Delta\sigma_{eff})^m]}\cdot(\frac{2}{2+m})\cdot a_o^{\frac{2+m}{2}}=$$

$$=\frac{2}{2+m}CUM_K^m\,\pi^{\frac{m}{2}}\frac{E[(\Delta\sigma_{eff})^{2m}]}{E[(\Delta\sigma_{eff})^m]}[(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t)^{\frac{2+m}{2-m}}-a_o^{\frac{2+m}{2}}]$$

Hence, $A(t)$ - a variance of crack length for the time t has the expression:

$$A(t)=\frac{2}{2+m}CUM_K^m\,\pi^{\frac{m}{2}}\frac{E[(\Delta\sigma_{eff})^{2m}]}{E[(\Delta\sigma_{eff})^m]}[(a_o^{\frac{2-m}{2}}+\frac{2-m}{2}\lambda UC\, M_K^m\,\pi^{\frac{m}{2}} E[(\Delta\sigma_{eff})^m]t)^{\frac{2+m}{2-m}}-a_o^{\frac{2+m}{2}}]\quad(29)$$

Step 4
Estimation of fatigue life of a component with the use of the function of failure risk

With the crack-length density function (25), one can determine the risk of failure which can occur in a component as a consequence of a crack developing up to a critical length in the time *t*:

$$Q(t)=\int_{a_{cr}}^{\infty} u(a,t)da\quad(30)$$

The critical crack length can be determined as follows:

$$a_{cr} = \frac{K_C^2}{M_k{}^2 \sigma_{cr}^2 \, \pi} \tag{31}$$

where:

$K_C = K_{IC} \cdot (1 + B \cdot \exp - (A \cdot g/g_0)^2)$,

$g_0 = 2{,}5 \cdot \left(K_{IC}/R_e\right)^2$,

symbols denote, respectively:

g - thickness of a sheet,

K_{IC} - material toughness,

R_e - yield stress,

A, B - material constants.

When the (required) $Q(t)_{req}$ is determined, some value of time should be found – such as to make the left side of equation (30) equal the right one. The value of "*t*" found in this way will be this sought fatigue life for the assumed level of the risk failure $Q(t)_{req}$. On the contrary, the reliability function $R(t)_{req} = 1 - Q(t)_{req}$ denotes the assumed level of probability that the current crack length a will not exceed a critical value of crack length a_{cr}.

3. EXPERIMENTAL VERIFICATION OF PROBABILISTIC METHOD FOR FATIGUE LIFE ESTIMATION

The probabilistic method was verified in this chapter by predicting the crack behavior and fatigue life estimation for aeronautical aluminum alloy sheet subjected to variable amplitude load program. Aluminum alloy 2024-T3 is used for lower skin aircraft wing structure. The material that was used in the experiment was 3mm thick sheet plated with pure aluminum on both sides (Alclad aluminum alloy sheets). The film of the plate had the thickness of 0,12mm. According to the manufacturer's certificate, the chemical composition of the 2024 alloy contains the following alloy elements: Cu (4,23%), Mg (1,37%), Mn (0,50%), Fe (0,18%), Zn (0,16%), Si (0,09%). The ranges of the mechanical properties of the alloy are presented in table 2.

Tab. 2.

	¤R_m, MPa	R_{02}, MPa	A, %
2024-T3	459-466	339-345	21,5-24,7

The T3 designation indicates the aluminum alloy's solution heat treating (495^{o}C, 50hours), cold rolling and natural aging at 195^{o}C. The specimens of 400 mm length and 100 mm width were cut out from the sheet. Subsequently, a through-thickness central hole of 5 mm in diameter, was cut inside each specimen. The hole had on each side a through-thickness saw cut of 2.5 mm length and an initial pre-crack of 2.5 mm length, the total length of the initial crack was equaled to 2a=20 mm. The hole served as crack initiator. Two series of specimens were made: LT-type specimens were cut out parallel while TL-type specimens were cut out perpendicularly to the sheet rolling direction.

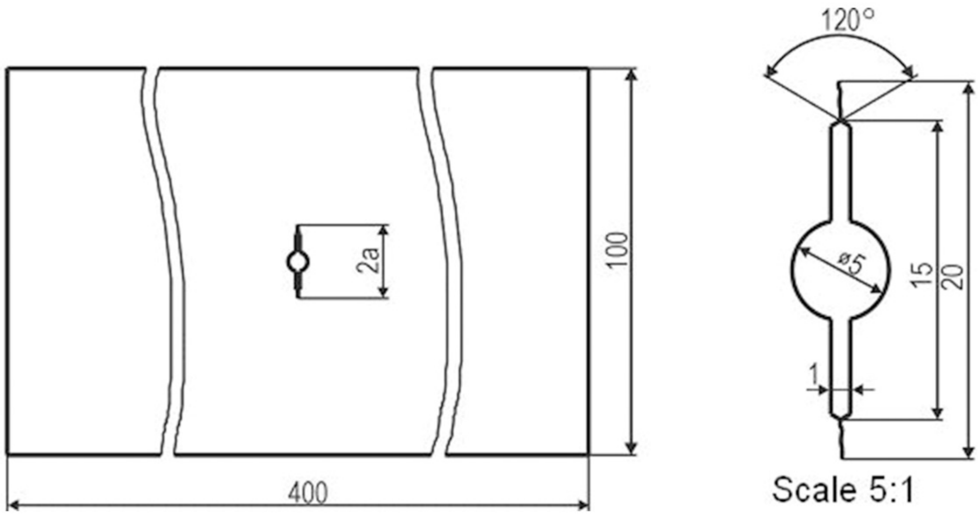

Fig. 1. Geometry of specimens

Fatigue tests on the specimens were carried out at room temperature under load control variable stress amplitude in accordance with the diagram given in Fig 2.

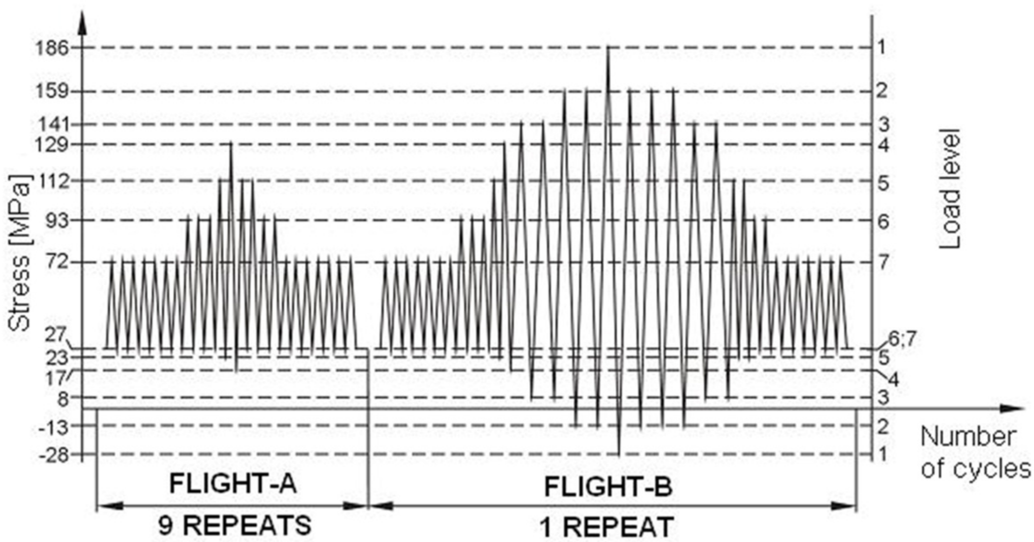

Fig. 2. Load spectrum LPL (flight by flight)

One sequence of the spectrum represents 10 aircraft flights and total number of cycles in the spectrum's sequence is 240. It consists of nine subspectrum called Flight-A and one subspectrum called Flight-B.

The characteristic of the above-mentioned variable amplitude load spectrum has been presented in table 3. The table contains maximum stress σ_{max}, minimum stress σ_{min} and mean stress $\sigma_{\acute{s}red}$ values for established 13 load levels. Furthermore, table contains ranges of stress values $\Delta\sigma$ together with probability of its appearance in used load spectrum.

Tab. 3.

Load level i	1	2	3	4	5	6	7
Number of cycles	1	5	4	10	30	50	140
σ_{max} [MPa]	186	159	141	129	112	93	72
σ_{min} [MPa]	-28	-13	8	17	23	27	27
$\sigma_{śred.}$ [MPa]	79	73	74,5	73	67,5	60	49,5
Amplitude [MPa]	107	86	66,5	56	44,5	33	22,5
Stress range $\Delta\sigma_i$ [MPa]	214	172	133	112	89	66	45
Load level contribution into load spectrum P_i	0,0042	0,0208	0,0167	0,0417	0,125	0,208	0,583

The effect of multiple overload/underload (OL/UL) cycles existing in the used spectrum on the crack growth rate in 2024-T3 aluminum alloy sheet was examined on the basis of the fatigue tests results. Exemplary courses of crack propagation rates in LT-type and TL-type specimens are illustrated on the following figures:

- Crack growth rate against number of load spectrum sequences – Fig.3;
- Crack length against number of load spectrum sequences – Fig.4;
- Crack growth rate against crack length – Fig.5.

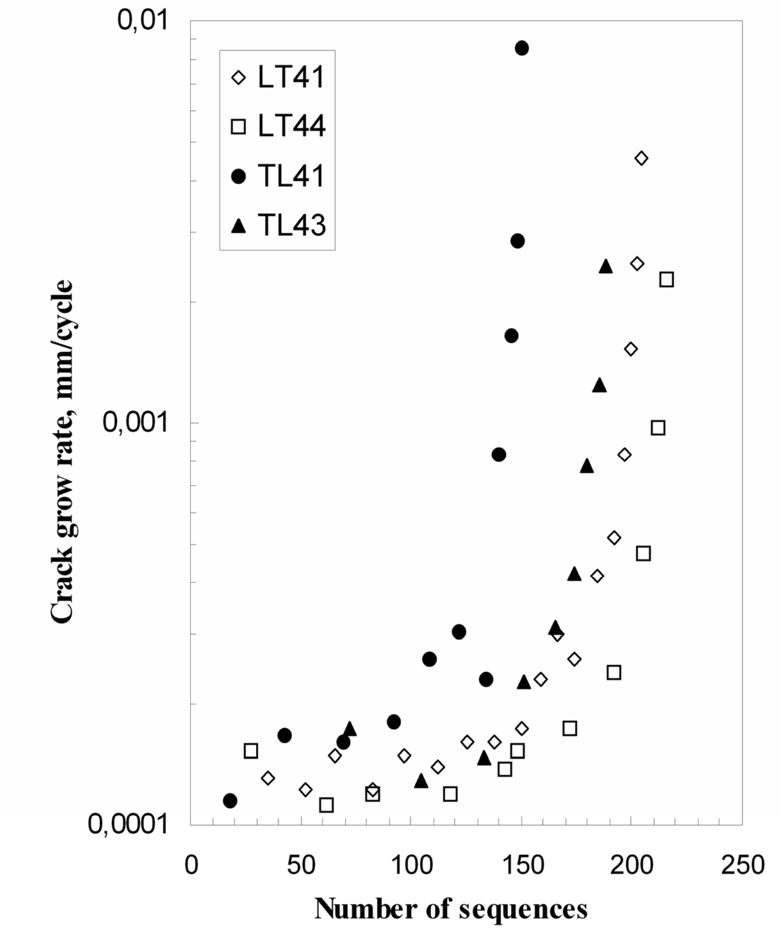

Fig. 3. Experimental curves for LT and TL samples of crack growth rate against number of load spectrum sequences

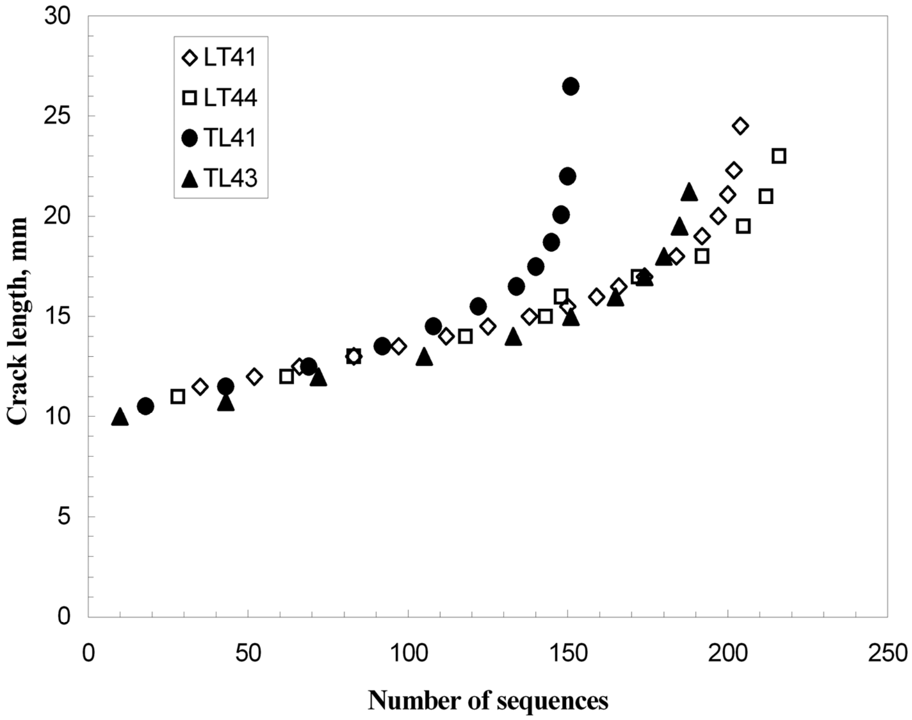

Fig. 4. Experimental curves for LT and TL samples of crack length against number of load spectrum sequences

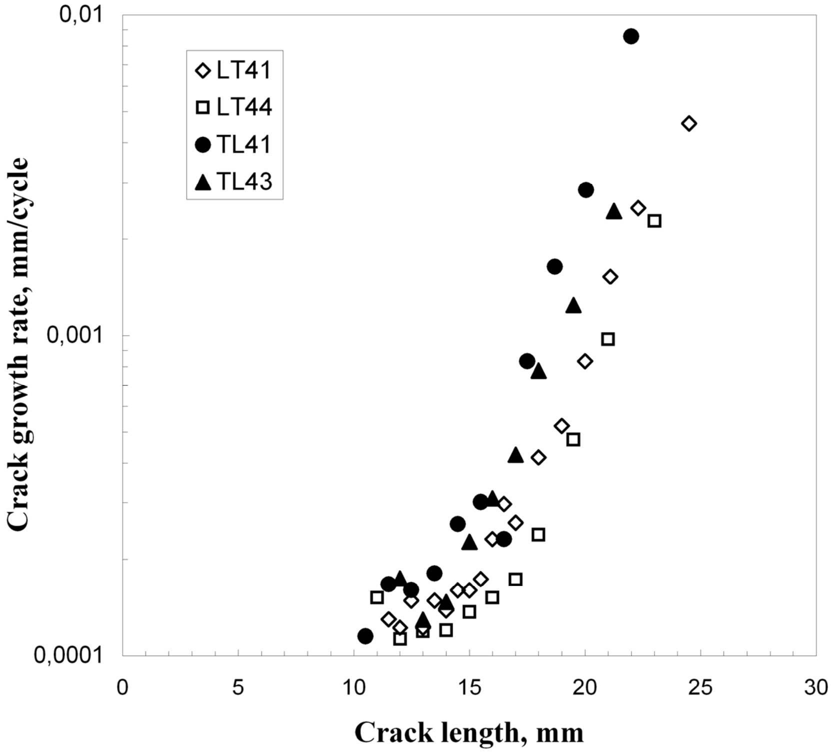

Fig. 5. Experimental curves for LT and TL samples of crack growth rate against crack length

The research revealed that real fatigue life of specimens has a range of values:

- $N = 160,\ 5\ sequences$ for TL samples;
- $N = 190\ldots210\ sequences$ for LT samples.

On the basis of experimental results the verification of the presented above probabilistic method for calculating the fatigue life was carried out and compared with experimental fatigue life.

Fatigue test results of 2024-T3 aluminum alloy sheet made possible to plot the curve of crack growth rate versus stress intensity factor and on the basis of this curve the coefficients of Paris formulae were calculated:

$m = 3{,}58$

$C = 3{,}4 \cdot 10^{-14}$

Data contained in table 3 made possible to calculate values of the following moments:

$$E[(\Delta\sigma)^m] = P_1(\Delta\sigma_1)^m + P_2(\Delta\sigma_2)^m + \ldots + P_L(\Delta\sigma_L)^m = 6{,}96 \cdot 10^6\,[MPa^m] \quad (32)$$

$$E[(\Delta\sigma)^{2m}] = P_1(\Delta\sigma_1)^{2m} + P_2(\Delta\sigma_2)^{2m} + \ldots + P_L(\Delta\sigma_L)^{2m} = 4{,}75 \cdot 10^{14}\,[MPa^{2m}] \quad (33)$$

Estimation of fatigue life was made for the crack growing from initial crack length $a_0 = 10\ mm$ to admissible crack length $a_d = 27\ mm$. In the next step the correction coefficient M_k was calculated (taking into consideration finite specimen dimensions) using empirical formula (34) [3] for the specified specimen's geometry of the CCT type (fig.1).

$$M_k = \left[1 - 0{,}025\left(\frac{a}{W}\right)^2 + 0{,}06\left(\frac{a}{W}\right)^4\right] \cdot \sqrt{\sec\left(\frac{\pi \cdot a}{2 \cdot W}\right)} \quad (34)$$

where W is one half width of a plate.

Next step concerned the equations that were used for calculating the average crack length $B(N)$ and the variance $A(N)$, respectively:

$$B(N) = [a_o^{\frac{2-m}{2}} + \frac{2-m}{2} C M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma)^m] N]^{\frac{2}{2-m}} - a_o \quad (35)$$

$$A(N) = \frac{2}{2+m} C M_K^m \pi^{\frac{m}{2}} \frac{E[(\Delta\sigma)^{2m}]}{E[(\Delta\sigma)^m]} [(a_o^{\frac{2-m}{2}} + \frac{2-m}{2} C M_K^m \pi^{\frac{m}{2}} E[(\Delta\sigma)^m] N)^{\frac{2+m}{2-m}} - a_o^{\frac{2+m}{2}}] \quad (36)$$

where: N is the number of load cycles.

As the result of calculation there were received the following functions:

$$B(N) = [0{,}162181 - 2{,}06 \cdot 10^{-6} \cdot N]^{-1{,}26582} - 10 \quad (37)$$

$$A(N) = 9{,}18 \cdot 10^{-6}[(0{,}162181 - 2{,}06 \cdot 10^{-6} \cdot N)^{-3{,}5316} - 616{,}595] \quad (38)$$

Finally, there was used the formula for estimating the risk of failure occurrence in a component as a consequence of a crack developing up to the length beyond admissible crack length a_d:

$$\bar{Q}(N)_{dop} = \int_{a_d}^{\infty} \frac{1}{\sqrt{2\pi A(N)}} e^{-\frac{(a-B(N))^2}{2A(N)}} da \quad (39)$$

For the assumed value of the risk of failure $\bar{Q}(N)^*_{dop} = 0{,}000001$ we try to find such a number of load cycles N, which fulfills the above equation (make the left side of equation (39) equal the right one).

As the result of the calculation, value N=41400 *cycles* is the searched fatigue life (expressed by the number of cycles) for the assumed level of failure's risk $\bar{Q}(N)^*_{dop} = 0{,}000001$.

The same value of fatigue life expressed by the number of load sequences takes value N=172,5 *sequences.*

For the presentation of the specimen's reliability course, fig. 6 describes its reliability $R(N) = 1 - Q(N)$ as a function of load sequences number.

Next, Fig. 7. presents the curve of predicted average crack length in the time $B(N)$ as the function of the number of load sequences obtained using equation (37).

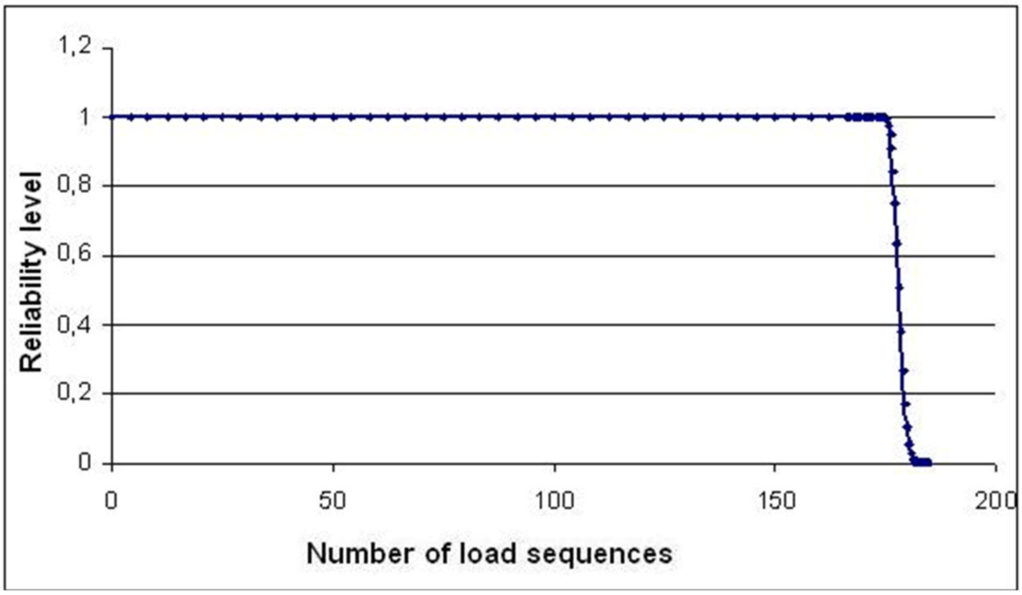

Fig. 6. Course of specimen's reliability function versus number of load sequences

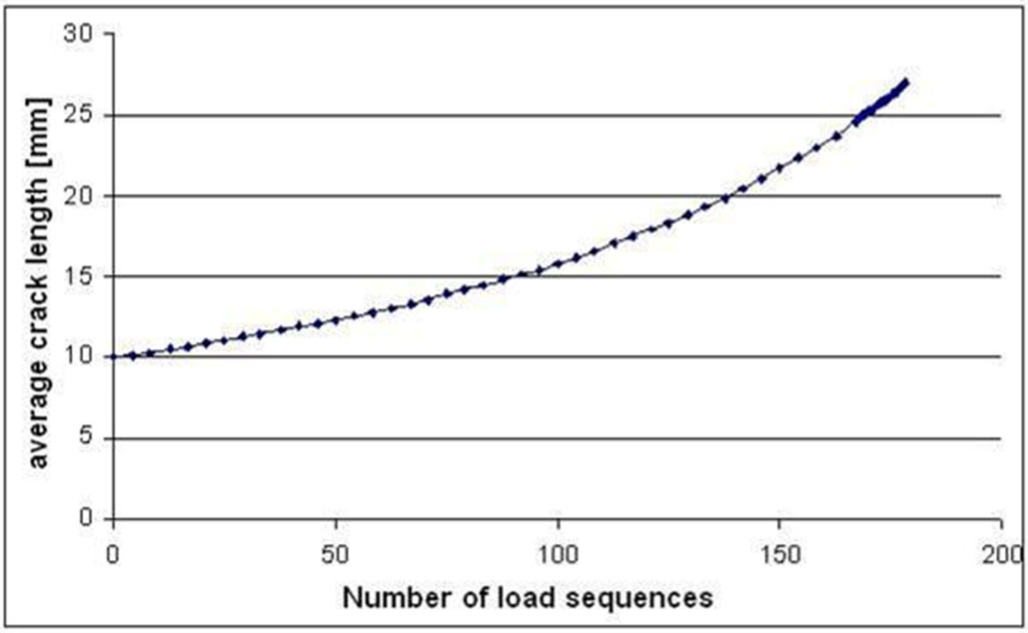

Fig. 7. Curve of predicted average crack length B(N) dependent on number of load sequences

The result of fatigue life calculation ($N = 172,5\ sequences$) proves that its value is contained in the range of the fatigue lives observed in the experiment ($N = 160,5 \ldots 210\ sequences$). On the basis of this verification one can put forward a proposal that the presented probabilistic method of fatigue life prediction under variable amplitude loading may be practically used for the structure's element fatigue life estimation.

4. SUMMARY

The probabilistic method presented in this paper facilitates a simplified description of fatigue crack growth under variable amplitude loading and the estimation of fatigue life. Fundamental to the description is a finite difference equation with the coefficients originated from the Paris formula, which models the dynamics of crack growth. The characteristic features of crack growth under overload-underload cycles existed in an exploitive loading were modelled by using the modified Willenborg retardation model. The presented probabilistic method has a good confirmation by experimental research of crack behaviour and fatigue life estimation for an aeronautical aluminium alloy sheet 2024-T3 subjected to variable amplitude load program. This method needs an extension over the crack initiation period.

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