

Optimal growth processes in a non-stationary Gale economy with a multilane production turnpike¹

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Abstract: The topic of the paper is relevant in the field of optimal growth theory and therefore might be seen as an intellectual underpinning for research and practice in the field of transition economies and sustainable long-time development as well. It refers to the papers Panek (2015a, 2018) devoted to asymptotic properties of optimal growth properties in the non-stationary Gale type economy with single and multi-lane turnpikes in which it was assumed that changing production technology converges in time with certain limits of technology. As far as the postulate of a non-stationary economy (here: technology change) is consistent with real processes, the hypothesis of the existence of some limiting technology may raise controversies and be difficult to verify.

In the paper, referring to the above mentioned publications and Panek (2014), a Gale-type economy with changing technology, multi-lane turnpike and time-increasing production efficiency, with no assumption concerning the existence of a limit technology will be examined.

Keywords: non-stationary Gale's economy, von Neumann's temporary equilibrium, multi-lane turnpike.

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Introduction

In mathematical economics a vast majority of the theorems focus on turnpike properties of optimal growth processes in stationary economies with constant technology and a single-lane turnpike.³ Attempts to prove the turnpike effect

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³ See e.g. (Gale, 1967; Jensen, 2012; Khan & Piazza, 2011; Majumdar, 2009; Makarov & Rubinov, 1977; McKenzie, 1976, 2005; Mowszowicz, 1969; Nikaido, 1968, ch. 4; Panek, 2003, ch. 5, 6; Takayama, 1985, ch. 7).

in non-stationary economies with changing technology are less frequent.⁴ A general analysis of the results recognizing a non-stationary Gale economy with a multi-lane turnpike and limit technology is included in articles by Panek (2017, 2018). The explicit assumption of production technology converging to a hypothetical limit technology may raise some objections, not to mention the fact that it is difficult to verify this convergence empirically and thus it was relaxed in this paper.

The rest of the paper is as follows. In sections 1 and 2 a non-stationary Gale-type model of the economy and define multilane production turnpike is presented. The main result of section 3 is a temporary von Neumann equilibrium theorem (Th. 1). In section 4 feasible and stationary growth processes in the economy are defined. Sections 5, 6 contain proofs of the so called “weak” (Th. 2) and “fast very strong” (Th. 3) turnpike theorems for the economy under investigation. The paper closes with some final remarks.

1. The model. Basic assumptions

The basic version of the presented model was introduced in Panek (2014). It is assumed that time t is discrete, $t = 0, 1, \dots$. In the considered economy there are n goods used up in production or produced in period t . By $x(t) = (x_1(t), \dots, x_n(t))$ a vector of goods used in period t (input vector) is denoted while $y(t) = (y_1(t), \dots, y_n(t))$ stands for a vector of goods produced in period t (output vector).⁵ If production technology in the economy enables to produce an output vector $y(t)$ from an input vector $x(t)$, it is said that the pair $(x(t), y(t))$ is a feasible production process in period t . By $Z(t) \subset R_+^{2n}$ the set of all feasible production processes in t . So $(x, y) \in Z(t)$ (or equivalently $(x(t), y(t)) \in Z(t)$) is denoted which means that in the economy in period t one can produce output vector y from input vector x . The set $Z(t)$ is called the production space (technology set) of the economy in period t . It is assumed that the production spaces $Z(t)$, $t = 0, 1, \dots$, satisfy the following conditions:

$$(G1) \quad \forall (x^1, y^1) \in Z(t) \forall (x^2, y^2) \in Z(t) \forall \lambda_1, \lambda_2 \geq 0 \left(\lambda_1 (x^1, y^1) + \lambda_2 (x^2, y^2) \in Z(t) \right)$$

(homogeneity and additivity of production processes).

$$(G2) \quad \forall (x, y) \in Z(t) \quad (x = 0 \Rightarrow y = 0)$$

(no land-of-Cocaigne condition).

⁴ See e.g. (Gantz, 1980; Joshi, 1997; Keeler, 1972; Panek, 2013, 2014, 2015a, b). They assume, that there exists a single-lane turnpike.

⁵ Coordinates of vectors $x(t)$, $y(t)$ measure inputs and outputs, respectively, and are expressed in physical units (kilograms, liters, meters, units, etc.).

$$(G3) \forall (x, y) \in Z(t) \forall x' \geq x \forall 0 \leq y' \leq y ((x', y') \in Z(t))^6$$

(costless waste possibility).

$$(G4) Z(t) \text{ is a closed subset of } R_+^{2n}$$

$$(G5) Z(0) \neq \emptyset \wedge Z(t) \subseteq Z(t+1)$$

(each production process $(x, y) \in Z(t)$ feasible in period t is also feasible in the next period).

Production spaces which satisfy the above conditions are closed convex cones in R_+^{2n} , with vertices at 0. If $(x, y) \in Z(t)$ and $(x, y) \neq 0$, then, according to (G2), $x \neq 0$.

The paper is only interested in non-trivial (non-zero) processes, that is, in processes $(x, y) \in Z(t) \setminus \{0\}$.

2. Technological and economic production efficiency. Multilane production turnpike

Let us fix a period t and a feasible process $(x, y) \in Z(t) \setminus \{0\}$. A non-negative number

$$\alpha(x, y) = \max\{\alpha \mid \alpha x \leq y\}$$

is called the technological efficiency rate of the process (x, y) in period t . If conditions (G1)-(G5) are satisfied, the function $\alpha(\cdot)$ is positively homogenous of degree zero on $Z(t) \setminus \{0\}$. Moreover, there exists

$$\alpha_{M,t} = \max_{(x,y) \in Z(t) \setminus \{0\}} \alpha(x, y) = \alpha(\bar{x}(t), \bar{y}(t)) \geq 0^7, \quad (1)$$

which is called the optimal efficiency rate in the non-stationary Gale economy in period t and, by (G5):

$$\alpha_{M,t+1} \geq \alpha_{M,t}. \quad (2)$$

The process $(\bar{x}(t), \bar{y}(t))$ that satisfies condition (1) is called an optimal production process in period t . Let us observe that any positive multiple of an optimal process $(\bar{x}(t), \bar{y}(t))$ is also an optimal process:

⁶ If $x, y \in R^n$, then $x \geq y$ means that $\forall i (x_i \geq y_i)$, in contrast to $x \geq y$ which means $x \geq y$ and $x \neq y$.

⁷ See, for example, Takayama (1985, Th. 6.A.1).

$$\forall \lambda > 0 \left(\alpha(\bar{x}(t), \bar{y}(t)) = \alpha(\lambda \bar{x}(t), \lambda \bar{y}(t)) = \alpha_{M,t} \right).$$

To exclude the unrealistic case of zero optimal technological production efficiency in any period t it is assumed that:

$$(G6) \exists (x, y) \in Z(0) (y > 0)$$

(in the initial period $t = 0$ the economy has access to technology which enables any good $i = 1, 2, \dots, n$ to be produced). With this assumption $\alpha_{M,0} > 0$ due to (2), ensures that $\forall t \geq 0$ ($\alpha_{M,t} > 0$). By Z_{opt} the set of all optimal production processes in t is denoted:

$$Z_{opt}(t) = \{(\bar{x}, \bar{y}) \in Z(t) \setminus \{0\} \mid \alpha(\bar{x}, \bar{y}) = \alpha_{M,t} > 0\}.$$

The sets $Z_{opt}(t)$, $t = 0, 1, \dots$, are convex cones in R_+^{2n} not containing 0 .⁸ If $(\bar{x}, \bar{y}) \in Z_{opt}(t)$, then according to (G1) and (G3) also

$$(\bar{x}, \alpha_{M,t} \bar{x}) \in Z_{opt}(t) \wedge (\bar{y}, \alpha_{M,t} \bar{y}) \in Z_{opt}(t).$$

The vector $s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|}$ is said to characterize the production structure in an optimal process $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)$.⁹ By $S(t)$ the set of production structures in all optimal processes in period t is denoted:

$$S(t) = \left\{ s(t) \mid \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|} \right) \right\}.^{10}$$

Assuming (G1)-(G6) sets $S(t)$, $t = 0, 1, \dots$, are non-empty, compact and convex.¹¹

Let us now consider any period t and an optimal production structure $s = s(t) \in S(t)$. The ray

$$N_s^t = \{\lambda s \mid \lambda > 0\} \subset R_+^n$$

is called a single-lane production turnpike (von Neumann ray) in the non-stationary Gale economy starting from period t . The set

⁸ The proof is as in Panek (2016, Th. 1).

⁹ Here and on, if $a \in R_+^n$, then $\|a\| = \sum_{i=1}^n a_i$.

¹⁰ Equivalently, $S(t) = \left\{ s(t) \mid \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{x}(t)}{\|\bar{x}(t)\|} \right) \right\}$.

¹¹ For a proof, see Panek (2016, Th. 2).

$$\mathbb{N}^t = \bigcup_{s \in S(t)} N_s^t = \{ \lambda s \mid \lambda > 0, s \in S(t) \}$$

is called a multi-lane production turnpike in the non-stationary Gale economy starting from period t . Observe that if $y \in \mathbb{N}^t$ and $\lambda > 0$, then $\lambda y \in \mathbb{N}^t$. If $y^1, y^2 \in \mathbb{N}^t$, so $s^1 = \frac{y^1}{\|y^1\|} \in S(t)$ and $s^2 = \frac{y^2}{\|y^2\|} \in S(t)$. Suppose $y = y^1 + y^2$. Then

$$s = \frac{y}{\|y\|} = \frac{y^1 + y^2}{\|y^1 + y^2\|} = \lambda_1 s^1 + \lambda_2 s^2,$$

where $\lambda_1 = \frac{\|y^1\|}{\|y^1 + y^2\|} > 0$, $\lambda_2 = \frac{\|y^2\|}{\|y^1 + y^2\|} > 0$, $\lambda_1 + \lambda_2 = 1$. The set $S(t)$ is convex, so $s \in S(t)$, and $y = y^1 + y^2 \in \mathbb{N}^t$. Hence each multi-lane turnpike $\mathbb{N}^t$ is a convex cone in R_+^n not containing 0.

□ **Lemma 1.** If in the non-stationary Gale economy satisfying conditions (G1)-(G6), in some period t , the input structure $\frac{x}{\|x\|}$ or production structure $\frac{y}{\|y\|}$ in a process $(x, y) \in Z(t) \setminus \{0\}$ differ from the turnpike structure,

$$\frac{x}{\|x\|} \notin S(t) \vee \frac{y}{\|y\|} \notin S(t),$$

then the technological efficiency of such process is less than the optimal efficiency rate:

$$\alpha(x, y) < \alpha_{M, t}.$$

Proof. Panek (2018; Lemma 1). ■

Equivalently, if in a process $(x, y) \in Z(t) \setminus \{0\}$

$$x(t) \notin \mathbb{N}^t \vee y(t) \notin \mathbb{N}^t$$

(input $x(t)$ or output $y(t)$ vector is off the multi-lane turnpike $\mathbb{N}^t$), so the technological efficiency of such a process is less than the optimal efficiency rate.

3. Von Neumann temporary equilibrium

Suppose $p(t) = (p_1(t), \dots, p_n(t)) \geq 0$ is price vector of goods in economy in period t and $(x(t), y(t)) \in Z(t) \setminus \{0\}$. The non-negative number¹²

¹² Here and on: if $a, b \in R^n$, then $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$.

$$\beta(x(t), y(t), p(t)) = \frac{\langle p(t), y(t) \rangle}{\langle p(t), x(t) \rangle}$$

(where $\langle p(t), x(t) \rangle \neq 0$) is called the economic efficiency rate of the process $(x(t), y(t))$ (at prices $p(t)$). If a price vector $\bar{p}(t) \geq 0$ and a production process $(\bar{x}(t), \bar{y}(t)) \in Z(t) \setminus \{0\}$ exist such that

$$\alpha_{M,t} \bar{x}(t) \leq \bar{y}(t), \quad (3)$$

$$\forall (x, y) \in Z(t) \left(\langle \bar{p}(t), y \rangle \leq \alpha_{M,t} \langle \bar{p}(t), x \rangle \right) \quad (4)$$

and

$$\langle \bar{p}(t), \bar{y}(t) \rangle > 0, \quad (5)$$

then the triplet $\{\alpha_{M,t}, (\bar{x}(t), \bar{y}(t)), \bar{p}(t)\}$ is said to be (characterize) temporary von Neumann equilibrium in the non-stationary Gale economy. The vector $\bar{p}(t)$ is called a temporary von Neumann equilibrium price vector in period t . From (3)-(5) it is concluded that

$$\alpha(\bar{x}(t), \bar{y}(t)) = \alpha_{M,t} > 0,$$

$$\beta(\bar{x}(t), \bar{y}(t), \bar{p}(t)) = \frac{\langle \bar{p}(t), \bar{y}(t) \rangle}{\langle \bar{p}(t), \bar{x}(t) \rangle} = \max_{(x, y) \in Z(t) \setminus \{0\}} \beta(x, y, \bar{p}(t)) = \alpha_{M,t}.$$

In the temporary von Neumann equilibrium (in period t) the economy attains not only the maximal technological efficiency rate $\alpha(\bar{x}(t), \bar{y}(t)) = \alpha_{M,t}$, but also the maximal economic efficiency rate $\beta(\bar{x}(t), \bar{y}(t), \bar{p}(t))$, which equals the technological efficiency rate $\alpha_{M,t}$ ¹³. The following condition (G7) together with (G1) (G6), ensures the existence of temporary von Neumann equilibrium in the non-stationary Gale economy in each period t :

$$(G7) \quad \forall t \quad \forall (x, y) \in Z(t) \setminus \{0\} \left(\alpha(x, y) < \alpha_{M,t} \Rightarrow \langle \bar{p}(t), y \rangle < \alpha_{M,t} \langle \bar{p}(t), x \rangle \right).$$

where $\bar{p}(t)$ satisfies condition (4).

Equivalently:

$$\forall t \quad \forall (x, y) \in Z(t) \setminus \{0\} \left(\alpha(x, y) < \alpha_{M,t} \Rightarrow \beta(x, y, \bar{p}(t)) < \alpha_{M,t} \right),$$

or

$$\forall t \quad \forall (x, y) \in Z(t) \setminus \{0\} \left(\beta(x, y, \bar{p}(t)) = \alpha_{M,t} \Rightarrow \alpha(x, y) = \alpha_{M,t} \right).$$

¹³ In temporary von Neumann equilibrium in t , the economic efficiency and the technological efficiency are equal and achieve their highest possible level (in a given period).

The condition states that in the Gale economy any process not attaining the maximal technological efficiency rate cannot attain the maximal economic efficiency rate.

□ **Theorem 1.** Under conditions (G1)-(G6) for each $t = 0, 1, 2, \dots$, there are prices $\bar{p}(t) \geq 0$ which satisfy (4). Moreover, if condition (G7) is fulfilled, then the triplet $\{\alpha_{M,t}, (\bar{x}(t), \bar{y}(t)), \bar{p}(t)\}$, for any process $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)$, is a temporary von Neumann equilibrium (satisfies conditions (3)-(5)). The optimal process $(\bar{x}(t), \bar{y}(t))$ as well as temporary equilibrium prices $\bar{p}(t)$ ($t = 0, 1, \dots$) are defined up to the structure.

Proof is the same the proof of Theorem 1 in Panek (2018)¹⁴. ■

4. Feasible and stationary growth processes

Let us fix a set of time periods $T = \{0, 1, \dots, t_1\}$, $t_1 < +\infty$. Named a horizon (of the economy). It is traditionally assumed that the economy is closed in the sense that the only source of inputs in the period $t + 1$ is the production (output) from the previous period t :

$$x(t+1) \leq y(t), t = 0, 1, \dots, t_1 - 1,$$

which due to (G3) leads to condition:

$$(y(t), y(t+1)) \in Z(t+1), t = 0, 1, \dots, t_1 - 1. \quad (6)$$

Let y^0 represent the production vector in period $t = 0$:

$$y(0) = y^0 \geq 0. \quad (7)$$

Every sequence of production vectors $\{y(t)\}_{t=0}^{t_1}$ satisfying conditions (6), (7) is called a (feasible) (y^0, t_1) – growth process (production trajectory) in the non-stationary Gale economy. The assumptions in this paper imply that such processes exist $\forall y^0 \geq 0 \quad \forall t_1 < +\infty$.

The interest is in the economy in which each output vector $\bar{y}(\check{t})$ of an optimal production process $(\bar{x}(\check{t}), \bar{y}(\check{t})) \in Z_{opt}(\check{t})$ in period $\check{t} < t_1$ is also input vector in some optimal production process $(\bar{x}(\check{t}+1), \bar{y}(\check{t}+1)) \in Z_{opt}(\check{t}+1)$ in the consecutive period:

$$\begin{aligned} \text{(G8)} \quad \forall \check{t} < t_1 \quad \forall (\bar{x}(\check{t}), \bar{y}(\check{t})) \in Z_{opt}(\check{t}) \quad \exists (\bar{x}(\check{t}+1), \bar{y}(\check{t}+1)) \in Z_{opt}(\check{t}+1) \\ (\bar{x}(\check{t}+1) = \bar{y}(\check{t})). \end{aligned}$$

¹⁴ After substituting $(\bar{x}, \bar{y}) \in Z_{opt}$ with $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)$, α_M with $\alpha_{M,t}$ and $\bar{p}$ with $\bar{p}(t)$.

By (6) and (G8),

$$\forall \check{t} < t_1 \forall s(\check{t}) \in S(\check{t}) \exists \left\{ \bar{y}(\check{t}) \right\}_{t=\check{t}}^{t_1} \left(\frac{\bar{y}(\check{t})}{\|\bar{y}(\check{t})\|} = s(\check{t}) \wedge (\bar{y}(t), \bar{y}(t+1)) \in Z(t+1), \bar{y}(t+1) = \alpha_{M, t+1} \bar{y}(t), t = \check{t}, \dots, t_1 - 1 \right). \quad (8)$$

If a sequence of production vectors $\{\bar{y}(t)\}_{t=\check{t}}^{t_1}$ satisfies condition (8), then

$$\bar{y}(t) = \left(\prod_{\theta=\check{t}+1}^t \alpha_{M, \theta} \right) \bar{y}(\check{t}), \quad t = \check{t} + 1, \dots, t_1. \quad (9)$$

Such a sequence is characterized by invariant production structure,

$$\forall t \in \{\check{t} + 1, \dots, t_1\} \left(\frac{\bar{y}(t)}{\|\bar{y}(t)\|} = \frac{\bar{y}(\check{t})}{\|\bar{y}(\check{t})\|} = s(\check{t}) = \text{const.} \right),$$

therefore it can be stated that it is a $(\check{t}, t_1)$ – stationary growth process (with constant production structure) at variable rate $\alpha_{M, t}$, $t = \check{t} + 1, \dots, t_1$. Each positive multiple of $(\check{t}, t_1)$ – stationary growth process (9) and the sum two such processes is also $(\check{t}, t_1)$ – a stationary process.

From (G8) $S(t) \subseteq S(t+1)$, i.e. $\mathbb{N}^t \subseteq \mathbb{N}^{t+1}$, $t = 0, 1, \dots, t_1 - 1$.¹⁵ is also obtained. The set (bundle of turnpikes)

$$\mathbb{N}^{t_1} = \bigcup_{t=0}^{t_1} \mathbb{N}^t.$$

is the greatest ('widest') multi-lane turnpike in the non-stationary Gale economy in horizon T . For $\check{t} = 0$ each stationary process $\{\bar{y}(t)\}_{t=0}^{t_1}$ of (9) satisfies the following condition:

$$\bar{y}(0) \in \mathbb{N}^0 \wedge \forall t \in \{1, \dots, t_1\} \left(\bar{y}(t) = \left(\prod_{\theta=1}^t \alpha_{M, \theta} \right) \bar{y}(0) \in \mathbb{N}^0 \subseteq \dots \subseteq \mathbb{N}^{t_1} \right),$$

so throughout all periods (starting from $\check{t} = 0$) it belongs to the multi-lane turnpike $\mathbb{N}^{t_1}$. For $\check{t} = 1$, each stationary process $\{\bar{y}(t)\}_{t=1}^{t_1}$ meets the following condition:

$$\bar{y}(1) \in \mathbb{N}^1 \wedge \forall t \in \{2, \dots, t_1\} \left(\bar{y}(t) = \left(\prod_{\theta=2}^t \alpha_{M, \theta} \right) \bar{y}(1) \in \mathbb{N}^1 \subseteq \dots \subseteq \mathbb{N}^{t_1} \right),$$

¹⁵ Due to technological development the number of von Neumann rays ('fast lanes') creating the multi-lane turnpike may increase.

so it belongs to the multi-lane turnpike $\mathbb{N}^{t_1}$ starting from period $t = 1$. Generally each $(\check{t}, t_1)$ – stationary process $\{\check{y}(t)\}_{t=\check{t}}^{t_1}$ ($\check{t} < t_1$) belongs to the multi-lane turnpike $\mathbb{N}^{t_1}$ from period t till the end of the horizon T . On the multi-lane turnpike the economy achieves its highest growth rate in each period of the functioning horizon.

5. Optimal growth processes. “Weak” turnpike effect

Suppose $u: R_+^n \rightarrow R_+^1$ denotes the utility function defined on production vectors in the last period of horizon T and fulfills the following conditions:

(G9) (i) $u(\cdot)$ is continuous, positively homogeneous of degree 1, concave and increasing,

(2i) $\exists a > 0 \forall t_1 \forall s \in S_+(1) (u(s) \leq a \langle \bar{p}(t_1), s \rangle)$ where $\bar{p}(t_1)$ is a von Neumann price vector in the final period t_1 , $S_+(1) = \{s \in R_+^n \mid \|s\| = 1\}$,

(3i) $\forall s \in S(t_1) (u(s) > 0)$.¹⁶

Condition (i) is standard, (2i) states that irrespective of the length of horizon T the utility function may be approximated from above by a linear form with vector of coefficients $a\bar{p}(t_1)$ proportional to a von Neumann price vector in the final period t_1 of horizon T .

Let us consider the following final state optimization problem (utility maximization of production obtained in the last period t_1 in horizon T):

$$\begin{aligned} & \max u(y(t_1)) \\ & \text{subject to (6), (7)} \\ & \text{(with fixed } y^0). \end{aligned}$$

Its solution is called a (y^0, t_1, u) – optimal growth process (production trajectory) and denoted as $\{y^*(t)\}_{t=0}^{t_1}$. Under the above assumptions there exists a solution $\forall y^0 \geq 0 \forall t_1 < +\infty$ (Panek, 2003, ch. 5, Th. 5.7).¹⁷

While proving turnpike theorems (Theorem 2 and 3) a significant role is played by the following lemma, which is a version Radner’s Lemma (1961) adapted to the specific character of the model of a non-stationary Gale economy.

□ **Lemma 2.** If conditions (G1)-(G7) are satisfied, then

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta_{\varepsilon, t_1} \in (0, \alpha_{M, t_1}) \forall t \in T \forall (x, y) \in Z(t) \setminus \{0\} : \\ d(x, \mathbb{N}^{t_1}) \geq \varepsilon \Rightarrow \beta(x, y, \bar{p}(t_1)) \leq \alpha_{M, t_1} - \delta_{\varepsilon, t_1}, \end{aligned} \quad (10)$$

¹⁶ Conditions (i)-(3i) are satisfied e.g. by some utility functions of the CES class.

¹⁷ In Panek (2003) the problem of maximizing production value in the last period T measured at von Neumann prices in Gale economy with time-invariant technology was discussed but the proof presented there is also applicable to an economy with changing technology and utility function (G9).

where

$$d(x, \mathbb{N}^{t_1}) = \inf_{x' \in \mathbb{N}^{t_1}} \left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| \quad (11)$$

is the (angular) distance of a vector x from the multi-lane turnpike $\mathbb{N}^{t_1}$.

Proof.¹⁸ If a process $(x, y) \in Z(t) \setminus \{0\}$ fulfills the hypothesis then each process $\lambda(x, y)$ with a $\lambda > 0$ fulfills it as well. Therefore while proving the lemma only consider feasible processes (x, y) from the set will be considered.

$$V_\varepsilon(t) = \{(x, y) \in Z(t) \mid \|x\| = 1 \wedge d(x, \mathbb{N}^{t_1}) \geq \varepsilon\}.$$

The distance (11) can be expressed equivalently as:

$$d(x, \mathbb{N}^{t_1}) = \inf_{s \in S(t_1)} f(x, s),$$

where $f(x, s) = \left\| \frac{x}{\|x\|} - s \right\|$, $s = \frac{x'}{\|x'\|} \in S(t_1)$. As $f \in C^0(R_+^n \setminus \{0\} \times S(t_1))$ and the set $S(t_1)$ is compact (see footnote 10), so

$$\forall x \geq 0 \exists \bar{s} \in S(t_1) \left(f(x, \bar{s}) = \inf_{s \in S(t_1)} f(x, s) \right). \quad (12)$$

Let us fix any $t \in T$ and $\varepsilon > 0$. We shall demonstrate that the set $V_\varepsilon(t)$ is compact (bounded and closed in R^{2n}).

(Boundedness) Let us assume that $(x^i, y^i) \in V_\varepsilon(t)$, $i = 1, 2, \dots$, and $\|(x^i, y^i)\| \rightarrow +\infty$. Since $\forall i (\|x^i\| = 1)$, then $\|y^i\| \rightarrow +\infty$. So, following (G1),

$$(\xi^i, \eta^i) = \left(\frac{x^i}{\|y^i\|}, \frac{y^i}{\|y^i\|} \right) \in Z(t), \quad i = 1, 2, \dots, \xi^i \rightarrow 0 \quad \text{and} \quad \|\eta^i\| = 1,$$

so there exists such a subsequence $\{\xi^{i_j}, \eta^{i_j}\}_{j=1}^\infty$, that $\lim_j (\xi^{i_j}, \eta^{i_j}) = (0, \bar{\eta})$ and $\|\bar{\eta}\| = 1$. Production space $Z(t)$ is a closed subset of R_+^{2n} , so $(0, \bar{\eta}) \in Z(t)$, which contradicts (G3). The contradiction leads to the conclusion that $V(\varepsilon)$ is a bounded set.

(Closedness) Suppose $(x^i, y^i) \in V_\varepsilon(t)$, $i = 1, 2, \dots$, and $(x^i, y^i) \xrightarrow{i} (\bar{x}, \bar{y})$. So $(\bar{x}, \bar{y}) \in Z(t)$ and $\|\bar{x}\| = 1$. We shall demonstrate that $d(\bar{x}, \mathbb{N}^{t_1}) \geq \varepsilon$. Following (12)

$$\forall i \exists s^i \in S(t_1) \left(f(x^i, s^i) = \inf_{s \in S(t_1)} f(x^i, s) = d(x^i, \mathbb{N}^{t_1}) \geq \varepsilon \right).$$

¹⁸ The proof is based on the proof of Theorem 5 in Panek (2016).

The set $S(t_1)$ is compact, so:

$$\exists \{s^{i_j}\}_{j=1}^{\infty} \left(s^{i_j} \rightarrow \bar{s} \in S(t_1) \wedge f(x^{i_j}, s^{i_j}) = \inf_{s \in S(t_1)} f(x^{i_j}, s) = d(x^{i_j}, \mathbb{N}^{t_1}) \geq \varepsilon \right).$$

Then $f(\bar{x}, \bar{s}) = d(\bar{x}, \mathbb{N}^{t_1}) \geq \varepsilon$. The set $V_{\varepsilon}(t)$ is thus closed in R_+^{2n} .

Condition **(G5)** entails in particular that $V_{\varepsilon}(t) \subseteq V_{\varepsilon}(t+1)$, $t = 0, 1, \dots, t_1 - 1$. Then condition **(G7)** (due to Lemma 1) leads to

$$\forall (x, y) \in V_{\varepsilon}(t) \left(0 \leq \langle \bar{p}(t_1), y \rangle < \alpha_{M, t_1} \langle \bar{p}(t_1), x \rangle \right),$$

that is,

$$\forall (x, y) \in V_{\varepsilon}(t) \left(\langle \bar{p}(t_1), x \rangle > 0 \right)$$

and therefore:

$$\forall (x, y) \in V_{\varepsilon}(t) \left(0 \leq \beta(x, y, \bar{p}(t_1)) = \frac{\langle \bar{p}(t_1), y \rangle}{\langle \bar{p}(t_1), x \rangle} < \alpha_{M, t_1} \right).$$

Function $\beta(\cdot, \cdot, \bar{p}(t_1))$ is continuous on a compact set $V_{\varepsilon}(t)$ (as a quotient of two linear functions with a non-zero function in denominator), so, according to the Weierstrass Theorem, a solution to the problem exists.

$$\max_{(x, y) \in V_{\varepsilon}(t)} \beta(x, y, \bar{p}(t_1)) = \beta_{\varepsilon, t} < \alpha_{M, t_1}.$$

The inclusion $V_{\varepsilon}(t) \subseteq V_{\varepsilon}(t+1)$ results in

$$\forall \varepsilon > 0 \forall t \in \{0, 1, \dots, t_1 - 1\} (\beta_{\varepsilon, t} \leq \beta_{\varepsilon, t+1} < \alpha_{M, t_1}).$$

Then

$$\exists \delta_{\varepsilon, t_1} \in (0, \alpha_{M, t_1}) \forall t \in T \forall (x, y) \in V_{\varepsilon}(t) \left(\beta(x, y, \bar{p}(t_1)) \leq \alpha_{M, t_1} - \delta_{\varepsilon, t_1} \right)$$

(it is enough to assume $\delta_{\varepsilon, t_1} = \alpha_{M, t_1} - \beta_{\varepsilon, t_1} > 0$) or equivalently:

$$\forall (x, y) \in V_{\varepsilon}(t) \left(\bar{p}(t_1), y \leq (\alpha_{M, t_1} - \delta_{\varepsilon, t_1}) \bar{p}(t_1), x \right). \quad \blacksquare$$

The lemma does not exclude a highly unrealistic case when for some $\varepsilon > 0$

$$\lim_{t_1} \delta_{\varepsilon, t_1} = 0,$$

that is when the economic efficiency of a production process over time converges to the maximum rate, even though the input structure in the process

permanently differs from the optimal (turnpike) structure by ε . To eliminate such a situation it is assumed that

$$(G10) \quad \forall \varepsilon > 0 \exists v_\varepsilon > 0 \forall t_1 \left(\frac{\delta_{\varepsilon, t_1}}{\alpha_{M, t_1}} \geq v_\varepsilon \right).$$

According to Lemma 2, $\delta_{\varepsilon, t_1} \in (0, \alpha_{M, t_1})$, that is, $v_\varepsilon \in (0, 1)$.

In previous papers devoted to the turnpike effect in non-stationary Gale economies it was emphasized how important regular technological development for a stable economic growth is, which in Neumann-Gale-Leontief's models of economic dynamics is expressed by means of technological production efficiency of economy and is reflected by the attained growth rate, see (Panek, 2015b). Now, along with the assumption of technology development (G5), which enables an increase in the technological efficiency rate, the following condition which eliminates rapid changes and fluctuations in technological production efficiency is also assumed and as a result, economic growth rate:

$$(G11) \quad \exists \rho > 0 \left(\lim_t \frac{\prod_{\theta=1}^t \alpha_{M, \theta}}{\alpha_{M, t}^t} \geq \rho \right).$$

To explain meaning of the condition let us denote $\Gamma_t = \frac{\prod_{\theta=1}^t \alpha_{M, \theta}}{\alpha_{M, t}^t}$. As $\alpha_{M, t+1} \geq \alpha_{M, t} > 0$ (see (2)), the sequence $\{\Gamma_t\}_{t=1}^\infty$ is non-increasing and bounded:

$$1 \geq \Gamma_t = \frac{\alpha_{M, 1}}{\alpha_{M, t}} \cdot \frac{\alpha_{M, 2}}{\alpha_{M, t}} \dots \frac{\alpha_{M, t}}{\alpha_{M, t}} \geq \frac{\alpha_{M, 1}}{\alpha_{M, t+1}} \cdot \frac{\alpha_{M, 2}}{\alpha_{M, t+1}} \dots \frac{\alpha_{M, t}}{\alpha_{M, t+1}} \cdot \frac{\alpha_{M, t+1}}{\alpha_{M, t+1}} = \Gamma_{t+1} > 0,$$

thus it has a limit $\bar{c} \geq 0$. (G11) requires $\bar{c} \geq \rho > 0$. It will be demonstrated that with this condition the (non-decreasing) sequence $\{\alpha_{M, t}\}_{t=0}^\infty$ is also bounded.

Indeed suppose $\gamma_{\theta, t} = \frac{\alpha_{M, \theta}}{\alpha_{M, t}}$, $1 \leq \theta \leq t$. Then

$$0 < \Gamma_t = \prod_{\theta=1}^t \gamma_{\theta, t} \leq \gamma_{1, t},$$

because

$$\forall \theta \in \{1, 2, \dots, t-1\} (0 < \gamma_{\theta, t} \leq \gamma_{\theta+1, t} \leq 1).$$

Let us assume that $\{\alpha_{M,t}\}_{t=0}^{\infty}$ is unbounded, $\lim_t \alpha_{M,t} = +\infty$. Then $\lim_t \gamma_{1,t} = 0$, i.e. $\lim_t \Gamma_t = 0$, despite **(G11)**. So a non-decreasing and bounded sequence $\{\alpha_{M,t}\}_{t=0}^{\infty}$ has a limit:

$$\lim_t \alpha_{M,t} = \alpha_M < +\infty.^{19} \quad (13)$$

The condition **(G11)** is fulfilled when starting from a period t the economy achieves on a multi-lane turnpike the growth rate for example

$$\alpha_{M,t} \geq e^{cq^t} \alpha_M > 0, \quad c < 0, q \in (0, 1)^{20}$$

or $\ln \frac{\alpha_{M,t}}{\alpha_M} \geq cq^t$. Indeed since

$$\ln \Gamma_t = \ln \frac{\prod_{\theta=1}^t \alpha_{M,\theta}}{\alpha_{M,t}^t} = \sum_{\theta=1}^t \ln \frac{\alpha_{M,\theta}}{\alpha_{M,t}} \geq \sum_{\theta=1}^t \ln \frac{\alpha_{M,\theta}}{\alpha_M},$$

so if $\ln \frac{\alpha_{M,t}}{\alpha_M} \geq cq^t$ then $\ln \Gamma_t \geq c \sum_{\theta=1}^t q^\theta$ that is

$$\lim_t \ln \Gamma_t \geq c \lim_t \sum_{\theta=1}^t q^\theta = c \sum_{\theta=1}^{\infty} q^\theta = \frac{cq}{1-q}.$$

Therefore $e^{\ln \Gamma_t} = \Gamma_t \geq e^{c \sum_{\theta=1}^t q^\theta}$, and (since $c < 0$):

$$\lim_t \Gamma_t = \bar{c} \geq e^{c \sum_{\theta=1}^{\infty} q^\theta} = e^{\frac{cq}{1-q}} = \rho \in (0, 1).$$

The last condition that is needed while proving the ‘weak’ multi-lane turnpike theorem in a Gale non-stationary economy simply says that there is at least one feasible growth process leading to the multi-lane turnpike:

(G12) There is a $(y^0, \check{t})$ – feasible process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t} < t_1$, such that

$$\hat{y}(\hat{t}) \in \mathbb{N}^{\hat{t}} \subseteq \mathbb{N}^{t_1}.$$

¹⁹ The case $\lim_t \alpha_{M,t} = +\infty$ is unrealistic. From the formal point of view, in the simplest one-good case economy with production spaces $Z(t) \subseteq Z(t+1) \subset R_+^2$, $t = 0, 1, \dots$, it implies that the closure $\text{cl} \left(\bigcup_{t=0}^{\infty} Z(t) \right) = R_+^2$. Then, in the limit, the condition **(G2)** is violated.

²⁰ Equivalently: $\alpha_{M,t} \in \left[\frac{\alpha_M}{e^{c_1 q^t}}, \alpha_M \right]$, where $c_1 = -c > 0$.

□ **Theorem 2.** If conditions **(G1)**–**(G12)** are fulfilled, then for any $\varepsilon > 0$ there exists a natural number k_ε that the number of periods when (y^0, t_1, u) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ satisfies the condition

$$d(y^*(t), \mathbb{N}^{t_1}) \geq \varepsilon \quad (14)$$

does not exceed k_ε . The number k_ε is independent of the horizon T length.

Proof. As $\forall t \in T(Z(t) \subseteq Z(t_1))$, so from the definition of the (y^0, t_1, u) – optimal growth process, following (4), (6), we get

$$\langle \bar{p}(t_1), y^*(t+1) \rangle \leq \alpha_{M, t_1} \langle \bar{p}(t_1), y^*(t) \rangle, \quad t = 0, 1, \dots, t_1 - 1, \quad (15)$$

where $\|\bar{p}(t_1)\| = 1$. Let us assume that in periods $\tau_1, \tau_2, \dots, \tau_k < t_1$ condition (14) is satisfied. Then, under Lemma 2, there exists $\delta_{\varepsilon, t_1} \in (0, \alpha_{M, t_1})$ such that

$$\langle \bar{p}(t_1), y^*(t+1) \rangle \leq (\alpha_{M, t_1} - \delta_{\varepsilon, t_1}) \langle \bar{p}(t_1), y^*(t) \rangle, \quad t = \tau_1, \dots, \tau_k. \quad (16)$$

From (15), (16) we get

$$\langle \bar{p}(t_1), y^*(t_1) \rangle \leq \alpha_{M, t_1}^{t_1 - k} (\alpha_{M, t_1} - \delta_{\varepsilon, t_1})^k \langle \bar{p}(t_1), y^0 \rangle,$$

which, after considering **(G9)**, leads to the following upper limit of production efficiency in the last period of horizon T ,

$$u(y^*(t_1)) \leq a \alpha_{M, t_1}^{t_1 - k} (\alpha_{M, t_1} - \delta_{\varepsilon, t_1})^k \langle \bar{p}(t_1), y^0 \rangle, \quad (17)$$

where a is a positive number. By **(G12)**, a $(y^0, \check{t})$ – feasible process $\{\check{y}(t)\}_{t=0}^{\check{t}}$ achieves the multi-lane turnpike in period $\check{t} < t_1$, $\check{s}(\check{t}) = \frac{\check{y}(\check{t})}{\|\check{y}(\check{t})\|} \in S(\check{t})$, and the sequence of output vectors $\{\check{y}(t)\}_{t=0}^{t_1}$,

$$\check{y}(t) = \begin{cases} \check{y}(t), & t = 0, 1, \dots, \check{t} \\ \sigma \prod_{\theta=\check{t}+1}^t \alpha_{M, \theta} \check{s}(\check{t}), & t = \check{t} + 1, \dots, t_1 \end{cases} \quad (18)$$

$(\sigma = \|\check{y}(\check{t})\| > 0)$ is a (y^0, t_1) – feasible growth process. Moreover,

$$u(y^*(t_1)) \geq u(\check{y}(t_1)) = \sigma \left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M, \theta} \right) u(\check{s}(\check{t})) > 0. \quad (19)$$

From (17), (19)

$$a\alpha_{M,t_1}^{t_1-k}(\alpha_{M,t_1} - \delta_{\varepsilon,t_1})^k \langle \bar{p}(t_1), y^0 \rangle \geq \sigma \left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta} \right) u(\check{s}(\check{t})) > 0,$$

is arrived at, that is,

$$\left(\frac{\alpha_{M,t_1} - \delta_{\varepsilon,t_1}}{\alpha_{M,t_1}} \right)^k \geq \frac{\sigma u(\check{s}(\check{t}))}{a \left(\prod_{t=1}^{\check{t}} \alpha_{M,t} \right) \bar{p}(t_1), y^0} \cdot \frac{\prod_{t=1}^{t_1} \alpha_{M,t}}{\alpha_{M,t_1}^{t_1}}.$$

By (G10), (G11)

$$1 > (1 - \nu_\varepsilon)^k \geq \frac{\rho \sigma c_{\min}}{ac_{\max} \prod_{t=1}^{\check{t}} \alpha_{M,t}} > 0,$$

$$c_{\min} = \min_{s \in S(\check{t})} u(s) > 0, \quad c_{\max} = \max_{\substack{p \geq 0 \\ \|p\|=1}} \langle p, y^0 \rangle > 0, \text{ that is}$$

$$k \ln(1 - \nu_\varepsilon) \geq \ln C,$$

where $C = \frac{\rho \sigma c_{\min}}{\prod_{t=1}^{\check{t}} \alpha_{M,t}} > 0$. Thus, since $0 < 1 - \nu_\varepsilon < 1$, we obtain

$$k \leq \frac{\ln C}{\ln(1 - \nu_\varepsilon)} = A.$$

It is enough to assume that k_ε is the smallest positive integer greater than $\max\{0, A\}$. ■

If it is assumed that in the (y^0, t_1, u) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ the initial output vector y^0 is positive, then condition (G12) in Theorem 2 is redundant.²¹

²¹ It is easy to demonstrate that in an economy with a positive initial production vector there a $(y^0, \check{t})$ – feasible growth process of the form (18) exists leading to a turnpike already in the first period (for $\check{t} = 1$).

6. One special case

While proving the ‘weak’ multi-lane turnpike theorem the assumption of the existence of a feasible path from the initial y^0 to the multi-lane turnpike $\mathbb{N}^{t_1}$ plays an important role. It may, of course, occur that there an optimal process exists which, in some period $\check{t} < t_1$, leads to the multi-lane turnpike, $y^*(\check{t}) \in \mathbb{N}^{\check{t}} \subseteq \mathbb{N}^{t_1}$. In Theorem 3 such a process is considered, but condition (G9)(2i) is strengthened to

(G9)(2i') $\exists a > 0 \forall t_1 \forall s \in S_+^n(1) \left(u(s) \leq a \langle \bar{p}(t_1), s \rangle \right)$. Moreover, if $y^*(\check{t}) \in \mathbb{N}^{\check{t}}$ in period $\check{t} < t_1$, then $u(s^*) = a \langle \bar{p}(t_1), s^* \rangle$ where

$$s^* = s^*(\check{t}) = \frac{y^*(\check{t})}{\|y^*(\check{t})\|} \in S(\check{t}) \subset S_+^n(1).^{22}$$

The proof of Theorem 3 becomes more transparent when the following facts

are considered. Suppose $\Gamma_t = \frac{\prod_{\theta=1}^t \alpha_{M\theta}}{\alpha_{M,t}^t}$ and for $t > \bar{t}$:

$$\Gamma_{\bar{t},t} = \frac{\prod_{\theta=\bar{t}+1}^t \alpha_{M\theta}}{\alpha_{M,t}^{t-\bar{t}}}.$$

As $\alpha_{M,t+1} \geq \alpha_{M,t}$ (see (2)), we get:

$$\Gamma_{\bar{t}+1,t} \geq \Gamma_{\bar{t},t} > 0, \quad (20a)$$

$$\Gamma_{\bar{t},t} \geq \Gamma_{\bar{t}+1,t} > 0 \quad (20b)$$

and

$$1 \geq \Gamma_{\bar{t},t} \geq \Gamma_t. \quad (20c)$$

□ **Proposition 1.** $\forall \bar{t} < +\infty \exists \lim_{t \rightarrow \infty} \Gamma_{\bar{t},t} \geq \rho > 0$.

Proof. The sequence $\{\Gamma_{\bar{t},t}\}_{t=\bar{t}+1}^{\infty}$ is monotone (non-increasing) and bounded (see (20b)), so it has a limit. By (20c) and (G11) the assertion. ■

²² The condition states that the linear form generated by vector $p = a\bar{p}(t_1)$ not only limits the utility from above but it also determines the plane tangent to the graph of the utility function along a single-lane turnpike (von Neumann ray) $N_{s^*}^i \{ \lambda s^* \mid \lambda > 0 \} \in \mathbb{N}^{t_1}$.

□ **Proposition 2.**

$$\forall \delta \in (0, 1) \exists \bar{t} < +\infty \forall t \geq \bar{t} \left(\lim_t \Gamma_{i,t} \geq 1 - \delta \right).^{23} \quad (21)$$

Proof. In accordance with this paper's notation the following is arrived at

$$\Gamma_t = \Gamma_{\bar{t}} \cdot \Gamma_{\bar{t},t} \cdot \left(\frac{\alpha_{M,\bar{t}}}{\alpha_{M,t}} \right)^{\bar{t}}$$

($t > \bar{t}$). The sequences $\{\Gamma_t\}_{t=1}^\infty$, $\{\Gamma_{\bar{t}}\}_{\bar{t}=1}^\infty$ converge to the same limit

$$\lim_t \Gamma_t = \lim_{\bar{t}} \Gamma_{\bar{t}} = \bar{c} \geq \rho > 0,$$

The sequence $\{\Gamma_{\bar{t},t}\}_{t=\bar{t}+1}^\infty$ is non-increasing and its limit is

$$\lim_t \Gamma_{\bar{t},t} = c_{\bar{t}} \geq \rho > 0$$

(see Proposition 1). Let $\varphi_{\bar{t},t} = \left(\frac{\alpha_{M,\bar{t}}}{\alpha_{M,t}} \right)^{\bar{t}}$, $t > \bar{t}$. Then

$$\forall \bar{t} \forall t > \bar{t} (0 < \varphi_{\bar{t},t} \leq 1). \quad (22)$$

Suppose that

$$\exists \delta \in (0, 1) \forall \bar{t} \exists \check{t} \geq \bar{t} (\lim_t \Gamma_{i,t} = c_{\bar{t}} < 1 - \delta).$$

Then

$$\forall \bar{t} \exists \check{t} \geq \bar{t} \left(\lim_t \varphi_{i,t} = \frac{1}{c_{\bar{t}}} > \frac{1}{1 - \delta} > 1 \right),$$

which contradicts (22) and completes the proof. ■

The following proposition is a simple consequence of Proposition 2 and monotonicity of the sequence $\{\Gamma_{i,t}\}_{t=i+1}^\infty$.

□ **Proposition 3.** $\forall \delta \in (0, 1) \exists \bar{t} < +\infty \forall t > \hat{t} \geq \bar{t} (\Gamma_{i,t} \geq 1 - \delta).$ ■

²³ As time unfolds ($\bar{t} \rightarrow +\infty$) the differences among growth rates achieved on the multi-lane turnpike decrease asymptotically to 0, in spite of the fact that the growth rates may increase asymptotically. In the economy considered the condition is satisfied if, for instance, $\alpha_{M,t} = e^{cq^t} \alpha_M > 0$, $c < 0$, $q \in (0, 1)$; see note 19.

□ **Theorem 3.** Suppose the Gale economy satisfies conditions (G1)–(G8), (G9) (i), (2i'), (3i), (G10), (G11). Then for any $\varepsilon > 0$ a positive integer $\bar{t}$ exists such that, if (y^0, t_1, u) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ reaches the multi-lane turnpike $\mathbb{N}^{t_1}$ in period $\check{t} \geq \bar{t}$ ($\check{t} < t_1$),

$$y^*(\check{t}) \in \mathbb{N}^{t_1},$$

then

$$\forall t \in \{\check{t} + 1, \dots, t_1 - 1\} \left(d(y^*(t), \mathbb{N}^{t_1}) < \varepsilon \right).$$

Proof. If (y^0, t_1, u) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ reaches the multi-lane turnpike $\mathbb{N}^{t_1}$ in period $\check{t} < t_1$, then the process $\{\check{y}(t)\}_{t=0}^{t_1}$,

$$\check{y}(t) = \begin{cases} y^*(t), & t = 0, 1, \dots, \check{t} \\ \sigma \prod_{\theta=\check{t}+1}^t \alpha_{M,\theta} s^*(\check{t}), & t = \check{t} + 1, \dots, t_1 \end{cases}$$

$(\sigma = \|y^*(\check{t})\| > 0, s^*(\check{t}) = \frac{y^*(\check{t})}{\|y^*(\check{t})\|} \in S(\check{t}))$ is (y^0, t_1) – feasible²⁴, and therefore

$$u(y^*(t_1)) \geq u(\check{y}(t_1)) = \sigma \left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta} \right) u(s^*(\check{t})) > 0. \quad (23)$$

On the other hand each (y^0, t_1, u) – optimal growth process satisfies condition (15), which leads to

$$\langle \bar{p}(t_1), y^*(t_1) \rangle \leq \sigma \alpha_{M,t_1}^{t_1-\check{t}} \langle \bar{p}(t_1), s^*(\check{t}) \rangle. \quad (24)$$

Let us fix any $\varepsilon > 0$. Contrary to the assertion, assume that for some $\tau \in \{\check{t} + 1, \dots, t_1 - 1\}$ it holds

$$d(y^*(\tau), \mathbb{N}^{t_1}) \geq \varepsilon.$$

According to Lemma 2 there exists $\delta_{\varepsilon, t_1} \in (0, \alpha_{M, t_1})$ such that

$$\langle \bar{p}(t_1), y^*(\tau+1) \rangle \leq (\alpha_{M, t_1} - \delta_{\varepsilon, t_1}) \langle \bar{p}(t_1), y^*(\tau) \rangle = \sigma (\alpha_{M, t_1} - \delta_{\varepsilon, t_1}) \langle \bar{p}(t_1), s^*(\tau) \rangle. \quad (25)$$

From (24), (25)

$$\langle \bar{p}(t_1), y^*(t_1) \rangle \leq \sigma \alpha_{M, t_1}^{t_1-\check{t}-1} (\alpha_{M, t_1} - \delta_{\varepsilon, t_1}) \langle \bar{p}(t_1), s^*(\check{t}) \rangle,$$

²⁴ It is an equivalent of the (y^0, t_1) – feasible process $\{\check{y}(t)\}_{t=0}^{t_1}$ defined by formula (18).

is obtained and hence, due to **(G9)(2i')**,

$$\begin{aligned} u(y^*(t_1)) &\leq \sigma \alpha_{M,t_1}^{t_1-\check{t}-1} (\alpha_{M,t_1} - \delta_{\varepsilon,t_1}) \langle \bar{p}(t_1), s^*(\check{t}) \rangle = \\ &= \sigma \alpha_{M,t_1}^{t_1-\check{t}-1} (\alpha_{M,t_1} - \delta_{\varepsilon,t_1}) u(s^*(\check{t})). \end{aligned} \quad (26)$$

By combining (23) and (26)

$$\sigma \alpha_{M,t_1}^{t_1-\check{t}-1} (\alpha_{M,t_1} - \delta_{\varepsilon,t_1}) u(s^*(\check{t})) \geq \sigma \left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta} \right) u(s^*(\check{t})) > 0,$$

is obtained, that is,

$$1 \geq \frac{\alpha_{M,t_1} \left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta} \right)}{(\alpha_{M,t_1} - \delta_{\varepsilon,t_1}) \alpha_{M,t_1}^{t_1-\check{t}}} > 0. \quad (27)$$

Consider $\delta \in \left(0, \frac{\delta_{\varepsilon,t_1}}{\alpha_M} \right)$, where α_M is the limit of the monotonic (non-decreasing) sequence $\{\alpha_{M,t}\}_{t=1}^{\infty}$:

$$\forall t (\alpha_{M,t+1} \geq \alpha_{M,t} > 0) \quad \text{and} \quad \lim_t \alpha_{M,t} = \alpha_M < +\infty \quad (28)$$

(see (13)). Suppose $\bar{t}$ is a time period satisfying (21) and corresponding to δ , and $\check{t} \geq \bar{t}$.²⁵ Then from (27), according to Proposition 3 inequality is obtained:

$$\frac{\alpha_{M,t_1} - \delta_{\varepsilon,t_1}}{\alpha_{M,t_1}} \geq \Gamma_{\check{t},t_1} \geq 1 - \delta > 1 - \frac{\delta_{\varepsilon,t_1}}{\alpha_M},$$

that is

$$1 - \frac{\delta_{\varepsilon,t_1}}{\alpha_{M,t_1}} > 1 - \frac{\delta_{\varepsilon,t_1}}{\alpha_M}.$$

The condition is met only if $\alpha_M < \alpha_{M,t_1}$ but this contradicts (28). The contradiction is a result of assuming that $d(y^*(\tau), \mathbb{N}^{t_1}) \geq \varepsilon$ is satisfied for at least one $\tau \in \{\check{t} + 1, \dots, t_1 - 1\}$. The proof is completed. ■

²⁵ Period $\bar{t}$ is dependent on ε , since, according to (21), it depends on the number $\delta \in \left(0, \frac{\delta_{\varepsilon,t_1}}{\alpha_M} \right)$.

Conclusions

The theorem locates itself between the ‘strong’ and ‘very strong’ versions of turnpike theorems in a non-stationary Gale economy with multi-lane turnpike. Similar or approximate results can be achieved in the case of at least two other input-output models of economic dynamics, these are the von Neumann and Leontief models. This indicates a possibility for further study.²⁶ The possible weakening of (G11) is an open question.

References

- Gale, D. (1967). On optimal development in a multi-sector economy. *Review of Economic Studies*, 34(1), 1-18.
- Gantz, D. (1980). A strong turnpike theorem for a nonstationary von Neumann-Gale production model. *Econometrica*, 48(7), 1977-1990.
- Grune, L., & Guglielmi, R. (2018). Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems. *SIAM Journal of Control Optimization*, 56(2), 1282-1302.
- Jensen, M. K. (2012). Global stability and the “turnpike” in optimal unbounded growth models. *Journal of Economic Theory*, 142(2), 802-832.
- Joshi, S. (1997). Turnpike theorems in nonconvex nonstationary environments. *International Economic Review*, 38(1), 225-248.
- Keeler, E. B. (1972). A twisted turnpike. *International Economic Review*, 13(1), 160-166.
- Khan, M. A., & Piazza, A. (2011). An overview of turnpike theory: Towards the discounted deterministic case. In S. Kusuoka & T. Maruyama (Eds.), *Advances mathematical economics* (vol. 14, pp. 39-68). Tokyo-Dordrecht-Heidelberg-London-New York: Springer.
- Majumdar, M. (2009). Equilibrium and optimality: Some imprints of David Gale. *Games and Economic Behavior*, 66(2), 607-626.
- Makarov, V. L., Rubinov, A. M. (1977). *Mathematical theory of economic dynamics and equilibria*. New York-Heidelberg-Berlin: Springer-Verlag.
- McKenzie, L. W. (1976). Turnpike theory. *Econometrica*, 44(5), 841-866.
- McKenzie, L. W. (2005). Optimal economic growth, turnpike theorems and comparative dynamics. In K. J. Arrow, M. D. Intriligator (Eds.), *Handbook of mathematical economics* (2nd ed., vol. 3, chapter 26). Stanford: Elsevier.
- Mowszowicz, S. M. (1969). Teoriemy o magistrali w modeliach Neumanna-Gale’a (słabaja forma). *Ekonomika i matemaiczeskije metody*, 5(6), 877-889.
- Nikaido, H. (1968). *Convex structures and economic theory*. New York: Academic Press.
- Panek, E. (2003). *Ekonomia matematyczna*. Poznań: Wydawnictwo Akademii Ekonomicznej.

²⁶ It is worth mentioning that research on turnpike effect is vivid and it continues on the ground of optimal control theory, see (Grune & Guglielmi, 2018; Trelat, Zhang & Zuazua, 2018; Zaslavski, 2015).

- Panek, E. (2013). „Słaby” i „bardzo silny” efekt magistrali w niestacjonarnej gospodarce Gale’a z graniczną technologią. *Przegląd Statystyczny*, 60(3), 291-304.
- Panek, E. (2014). Niestacjonarna gospodarka Gale’a z rosnącą efektywnością produkcji na magistrali. *Przegląd Statystyczny*, 61(1), 5-14.
- Panek, E. (2015a). A turnpike theorem for a non-stationary Gale economy with limit technology. A particular case. *Economics and Business Review*, 1(15), 3-13.
- Panek, E. (2015b). Zakrzywiona magistrala w niestacjonarnej gospodarce Gale’a (cz. 1). *Przegląd Statystyczny*, 62(2), 149-163.
- Panek, E. (2016). Gospodarka Gale’a z wieloma magistralami. „Słaby” efekt magistrali. *Przegląd Statystyczny*, 63(4), 355-374.
- Panek, E. (2017). „Słaby” efekt magistrali w niestacjonarnej gospodarce Gale’a z graniczną technologią i wielopasmową magistralą. W: D. Appenzeller (red.), *Matematyka i informatyka na usługach ekonomii* (s. 94-110). Poznań: Wydawnictwo Uniwersytetu Ekonomicznego.
- Panek, E. (2018). Niestacjonarna gospodarka Gale’a z graniczną technologią i wielopasmową magistralą produkcyjną. „Słaby”, „silny” i „bardzo silny” efekt magistrali. *Przegląd Statystyczny*, 65(4), 373-393.
- Radner, R. (1961). Path of economic growth that are optimal with regard to final states:
A turnpike theorem. *Review of Economic Studies*, 28(2), 98-104.
- Takayama, A. (1985). *Mathematical economics*. Cambridge: Cambridge University Press.
- Trelat, E., Zhang, C., & Zuazua, E. (2018). Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces. *SIAM Journal of Control and Optimization*, 56(2), 1222-1252.
- Zaslavski, A. J. (2015). *Turnpike theory of continuous-time linear optimal control problems*. Cham-Heidelberg-New York-Dordrecht-London: Springer.