sciendo

Civil and Environmental Engineering
Vol. 14, Issue 2/2018, 153-158, DOI: 10.2478/cee-2018-0020

STABILITY AND PHASE PORTRAITS FOR SIMPLE DYNAMICAL SYSTEMS

Mária KÚDELČÍKOVÁ[1,*], Eva MERČIAKOVÁ[1]

[1] Department of Structural Mechanics and Applied Mathematics, Faculty of Civil Engineering, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia.
[*] corresponding author: maria.kudelcikova@fstav.uniza.sk.

Abstract

In structural dynamics models of mechanical oscillator and vibration analysis are of great importance. In this article motion of mechanical oscillator is modelled using second order linear autonomous differential systems. Stability of such 1 DOF models is investigated with respect to the coefficients of systems. Phase portraits for various cases are displayed and the character of fixed points is described.

Keywords:

Dynamical system;
Structural dynamics;
Stability;
Eigenvalues;
Phase portrait.

1 Introduction

The theory and applications of differential equations and their systems play an important role in modern dynamics. Such equations are mathematical models of various real-life phenomena, e.g. in population dynamics, ecology, medicine, economics, natural sciences, and last but not least in the structural dynamics when studying vibrations of structures that interact with fluid flows: bridges, buildings, airplanes. At a certain flow speed, increasing oscillations may be triggered. A famous example was the collapse of the Tacoma-Narrows suspension bridge near Seattle in 1940 under a moderate wind speed of about 40 mph. For this reason the study of the stability and long-time behaviour is very important.

1.1 Mathematical background

Informally we can state that some structure is stable at an equilibrium position (fixed point) if it returns to that position upon being disturbed by an extraneous action.

From the mathematical point of view we consider dynamical autonomous system of differential equations

$$
\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2, \dots, y_n),\\
\dot{y}_2 &= f_2(y_1, y_2, \dots, y_n),\\
&\vdots\\
\dot{y}_n &= f_n(y_1, y_2, \dots, y_n),
\end{aligned}
\tag{1}
$$

or in vector form $\dot{y} = f(y)$ where the vector function f is defined on some domain $\Omega \in R^n$. Domain Ω is called phase space and implicitly contained variable t is time. Functions on right-hand sides of (1) are continuous; partial derivatives $\delta f_i/\delta y_k$, i, k = 1, 2, ..., n exist and are continuous, too.

These suppositions guarantee that through each point in Ω it passes unique solution $y(t) = (y_1(t), y_2(t), \dots y_n(t))$. A solution $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots \varphi_n(t))$ to system (1) satisfying initial conditions $\varphi(t_0) = \varphi^0$ is called (Ljapunov) stable on the interval $[t_0,\infty)$ if for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for each solution to (1), when initial values satisfy inequalities $|y_i(t_0) - \varphi_i^0| < \delta$, i = 1, 2, ... n then it holds $|y_i(t) - \varphi_i(t)| < \varepsilon$, i = 1, 2, ... n for all $t \geq t_0$. A solution $\varphi(t)$ is called asymptotically stable if it is stable and it holds $lim_{t\to\infty}|y_i(t) - \varphi_i(t)| = 0$, i = 1, 2, ... n. Geometrical meaning is that solutions "close" at the beginning (for $t = t_0$) remain "close" also for all values $t \geq t_0$. Constant vector $y^* = (y_1^*, y_2^*, \dots, y_n^*) \in \Omega$ such that $f_i(y^*) = 0$, i = 1, 2, ... n is called equilibrium or fixed point. We note that $y = y^*$ is also a solution to (1). This problematics is very well described in [1].

In the following we deal with homogenous second order linear system of differential equations with constant coefficients. It is the simplest case of dynamical system and the interpretation of solutions in plane is possible. Despite of its simplicity it can be used as a model in many applications and it is important to study behaviour of its solutions. They can be different in dependence of system coefficients. Particular solution to system (1) can be understood as a graph of a function in domain R x Ω or as a curve in Ω given by parametric equations $y = y(t)$. Such curve is called trajectory or phase portrait of solution. Set of phase portraits of all solutions is called phase portrait of system.

Every second order linear differential equation can be rewritten as a system of two first order linear differential equations as follows. Let us consider the general equation

$$a\,\ddot{x}(t) + b\,\dot{x}(t) + c\,x(t) = 0, \tag{2}$$

where a, b, c are constants. If we define $x = x_1$ and $\dot{x}_1 = x_2$ then equation (2) comes into system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{c}{a}\,x_1 - \frac{b}{a}\,x_2, \end{aligned} \tag{3}$$

which can be written in standard form

$\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x}$,

where $\mathrm{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$ is constant matrix.

Stability and behaviour of solutions of such system can be determined using eigenvalues of A. Eigenvalues $\lambda_{1,2}$ can be found by solving the characteristic equation, i.e. $|\mathrm{A} - \lambda \mathrm{I}| = 0$. If all real parts of eigenvalues are negative, equilibrium is asymptotically stable. If there exists at least one eigenvalue with positive real part, equilibrium is unstable. In our case characteristic equation is

$$|\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \tag{4}$$

and its roots - eigenvalues are $\lambda_{1,2} = \frac{-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4\frac{c}{a}}}{2}$.

It is visible that stability, character of fixed points and phase portraits depend on constants a, b, $c \in R$. According to eigenvalues we can classify the simple fixed points. According to [2], if eigenvalues are real and different, we have three possible cases: if $\lambda_1 > 0 > \lambda_2$, fixed point is called saddle, if $\lambda_1 > \lambda_2 > 0$, it is called unstable node, and if $0 > \lambda_1 > \lambda_2$, it is stable node. If there exists only single eigenvalue, i.e. $\lambda_1 = \lambda_2$, equilibrium is called unstable improper node, if $\lambda_1 > 0$ and stable improper node, if $\lambda_1 < 0$. If eigenvalues are complex, i.e. $\lambda_{1,2} = \alpha \pm i\beta$, $\beta > 0$ equilibrium is called unstable focus, if $\alpha > 0$, stable focus, if $\alpha < 0$, and centre, if $\alpha = 0$.

2 Stability of dynamical systems with one degree of freedom

In order it would be possible to analyse dynamics of some complicated mechanical systems [3], first it is essential to understand the properties of the simplest possible mechanical system – a system with one degree of freedom [4]. In this part we will focus on vibrations, mathematically, on the stability and phase portraits of solutions to second order linear autonomous systems with constant coefficients modelling behaviour of undamped and damped mechanical oscillator.

2.1 The undamped mechanical oscillator

The motion of undamped mechanical oscillator is represented by the single rigid mass point attached to a spring displayed in Fig.1.

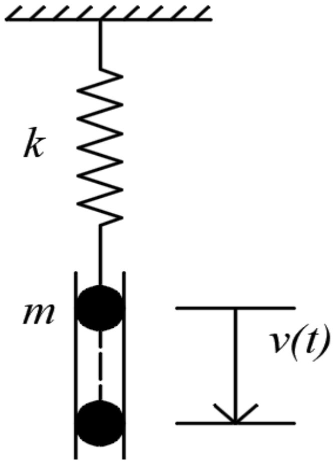

Fig. 1: Undamped mechanical oscillator.

It is assumed that the mass m causes vertical translation movements; the spring does not have any mass but stiffness k. No other external force is considered. The second order differential equation describing this motion is

$$m\ddot{v}(t) + kv(t) = 0.$$

It is the equation of type (2) with coefficient $b = 0$. Using substitution $v = x$ and $\dot{x} = y$ we come to homogeneous differential system of type (3) with zero equilibrium in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{5}$$

Characteristic equation for this system is according to (4) $\lambda^2 + k/m = 0$. Eigenvalues are complex with zero real parts $\lambda_{1,2} = \pm\, \mathrm{i}.\sqrt{k/m}$. Equilibrium is stable but not asymptotically stable. General solution to system (5) can be written in the form:

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \;\; y(t) = -C_1\sin(\omega t) + C_2\cos(\omega t), \;\; \omega = \sqrt{\frac{k}{m}}. \tag{6}$$

where $\omega[rad.s^{-1}]$ is the angular frequency and constants C_i, i = 1, 2 are determined by initial conditions of the oscillation. The sum of square roots of equations in (6) is $x^2 + y^2 = C_1^2 + C_2^2 > 0$ Phase trajectories are concentric circles, centred at the origin, phase portrait is displayed in Fig. 2 and equilibrium is called centre.

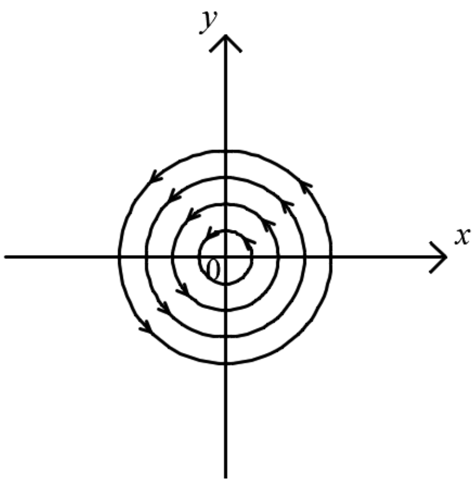

Fig. 2: Centre.

2.2 The damped mechanical oscillator

The equation of motion of damped oscillator is second order differential equation with constant positive coefficients of type (2) in the form

$$m\ddot{v}(t) + b\dot{v}\ (t) + kv(t) = 0.$$

Coefficient b represents a damping force and it is conjunction of velocity and viscose linear damping coefficient. Graphically, it is also represented by the spring and mass point as displayed in Fig. 3.

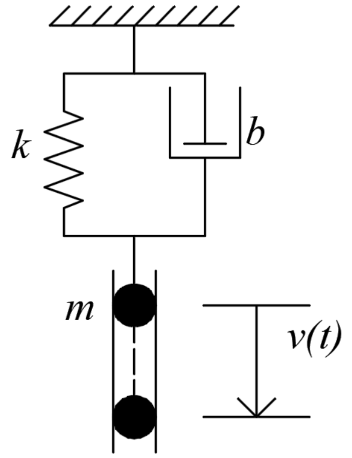

Fig. 3: Damped mechanical oscillator.

Using the same substitution as in previous case we obtain second order linear system of type (3) in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x - \frac{b}{m}y. \end{aligned} \tag{7}$$

Its characteristic equation is in accordance with (4) $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$ and eigenvalues are

$$\lambda_{1,2} = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2}. \tag{8}$$

Now we will focus on character of fixed point (0,0) in dependence on the values of system coefficients m, b and k. For their different values, phase portraits of system will be displayed and stability of trivial solution will be determined. Three cases depending on discriminant in (8) will be studied.

2.2.1 Critically damped oscillation

This oscillation occurs when $b = b_{cr} = 2\sqrt{km}$ hence value of damping coefficient is equal to value of critical damping coefficient, discriminant in (8) is equal zero. Since all coefficients must be positive, there exists single real negative eigenvalue $\lambda_1 = -b/(2m)$ trivial solution is asymptotically stable, equilibrium is stable improper node and the phase portrait is displayed in Fig. 4.

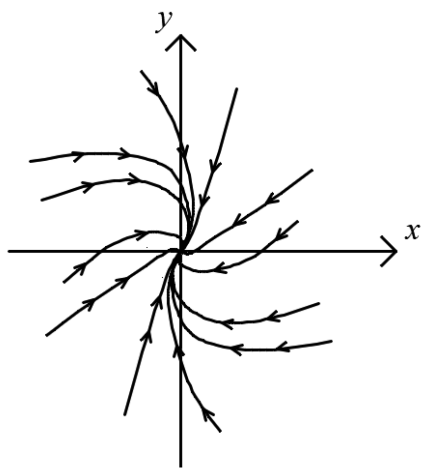

Fig. 4: Stable improper node.

We can consider concrete values of coefficients: example: m = 78.9 kg, k = 998667Nm^{-1}, and $b = b_{cr}$ = 17753.3 kg.s^{-1}. Then $\lambda_1 = -112.5 < 0$.

2.2.2 Overdamped oscillation

When $b > b_{cr}$, i.e. the value of damping coefficient is higher than critical damping coefficient, we speak about overdamped oscillation. Discriminant in (8) is positive, there exist two real negative and different eigenvalues, trivial solution is asymptotically stable, equilibrium is called stable node, the phase portrait is displayed in Fig. 5.

Example of concrete coefficients: m = 78.9 kg, k = 998667Nm^{-1}, and b = 18000 kg.s^{-1}, b_{cr}= 17753.3 kg.s^{-1}. Different negative eigenvalues are $\lambda_1 = -95.247$, $\lambda_2 = -132.890$.

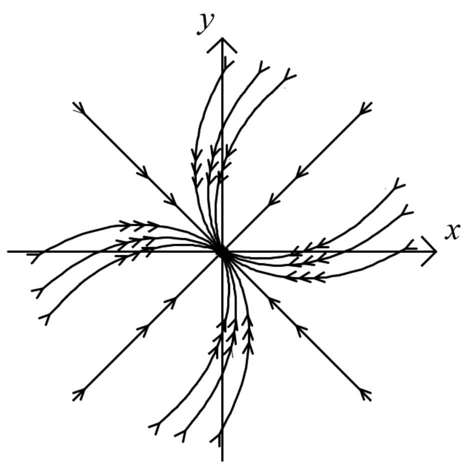

Fig. 5: Stable node.

2.2.3 Underdamped oscillation

If $b < b_{cr}$, i.e. the value of damping coefficient is lower than critical damping coefficient, the oscillation is underdamped. In this case, discriminant in (8) is negative, there exist two eigenvalues which are complex conjugates, their real parts are negative, trivial solution is asymptotically stable, equilibrium is called stable focus, the phase portrait is displayed in Fig. 6.

Example: m = 78.9 kg, k = 998667Nm^{-1}, and b = 1950 kg.s^{-1}, b_{cr} = 17753.3 kg.s^{-1}. Complex eigenvalues with negative real parts are: $\lambda_{1,2} = -12.357 \pm 111.824i$.

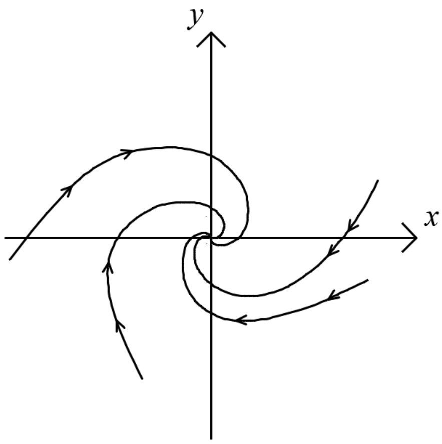

Fig. 6: Stable focus.

3 Conclusion

In this paper the connection between mathematical stability theory and vibration problems in structural dynamics was presented. Simple models of mechanical oscillator were studied in undamped, critically damped, overdamped and underdamped cases. Motions of mechanical oscillator were modelled using second order linear homogenous autonomous systems of differential equations. Phase portraits of systems were displayed for each case with respect to various system coefficients, especially damping coefficient, and character of equilibrium was described. In undamped case trivial solution is stable, but not asymptotically, phase portrait is formed by concentric circles and equilibrium is called centre. In other cases trivial solution is asymptotically stable. If damping coefficient is equal to critical damping, equilibrium is called stable improper node, if this coefficient is greater than critical damping, equilibrium is stable node, and if it is smaller, equilibrium is called stable focus. Knowledge of behaviour of simple dynamical systems can enable to study more complicated systems.

Acknowledgement

This work was supported by the Grant National Agency VEGA No. 1/0005/16.

References

[1] DIBLÍK, J. – RŮŽIČKOVÁ, M.: Ordinary Differential Equations. EDIS Žilina, 2008 (in Slovak).
[2] BOWTELL, G.: Dynamical Systems. City University, London, 2011.
[3] KUCHÁROVÁ, D. – MELCER, J.: Dynamics of Structures. EDIS Žilina, 2000 (in Slovak).
[4] LARSONNEUR, R.: Modelling and analysis of Dynamic Mechanical Systems. Winterthur, 2006.