

On Almost Complete Caps in $PG(N, q)$

Alexander A. Davydov¹, Stefano Marcugini², Fernanda Pambianco²

¹Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Bol'shoi Karetnyi pereulok 19, Moscow 127051, Russian Federation

²Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, Perugia 06123, Italy

E-mails: adav@iitp.ru stefano.marcugini@unipg.it fernanda.pambianco@unipg.it

Abstract: We propose the concepts of almost complete subset of an elliptic quadric in the projective space $PG(3, q)$ and of almost complete cap in the space $PG(N, q)$, $N \geq 3$, as generalizations of the concepts of almost complete subset of a conic and of almost complete arc in $PG(2, q)$. Upper bounds of the smallest size of the introduced geometrical objects are obtained by probabilistic and algorithmic methods.

Keywords: Projective space, complete cap, complete arc, almost complete cap, almost complete arc.

1. Introduction

Let $PG(N, q)$ be the N -dimensional projective space over the Galois field \mathbb{F}_q of order q . A cap in $PG(N, q)$ is a set of points no three of which are collinear. An n -cap of $PG(N, q)$ is complete if it is not contained in an $(n + 1)$ -cap of $PG(N, q)$. Caps in $PG(2, q)$ are called also arcs. A point P of $PG(N, q)$ is covered by a cap $\mathcal{K} \subset PG(N, q)$ if P lies on a bisecant of \mathcal{K} . The space $PG(N, q)$ contains $\theta_{N,q} = \frac{q^{N+1} - 1}{q - 1}$

points.

An n -arc in $PG(N, q)$ with $n > N + 1$ is a set of n points such that no $N + 1$ points belong to the same hyperplane of $PG(N, q)$. An n -arc of $PG(N, q)$ is complete if it is not contained in an $(n + 1)$ -arc of $PG(N, q)$. In $PG(N, q)$ with $2 \leq N \leq q - 2$, a normal rational curve is any $(q + 1)$ -arc projectively equivalent to the arc $\{(1, t, t^2, \dots, t^N) : t \in \mathbb{F}_q\} \cup \{(0, \dots, 0, 1)\}$.

For an introduction to the spaces on finite fields, see [8-11] and the references therein.

The concept of *almost complete subset of a fixed irreducible conic in the plane* $PG(2, q)$ is considered in [12], see also [3, 14] and the references therein.

Definition 1. In $\text{PG}(2, q)$, an almost complete subset of a fixed irreducible conic is a proper subset of the conic covering all the points of $\text{PG}(2, q)$ except for the remaining points of the conic and its nucleus if q is even.

Almost complete subsets of conics are useful for the classical problems of completeness of normal rational curves and extendability of generalized doubly-extended Reed-Solomon codes.

Let $t(q)$ be the smallest size of an almost complete subset of a conic in $\text{PG}(2, q)$. Let an $[n, k, d]_q$ code be a q -ary linear code of length n , dimension k , and minimum distance d .

In [14] it is proved that under the condition

$$3 \leq N \leq q + 2 - t(q),$$

every normal rational curve in $\text{PG}(N, q)$ is a complete $(q + 1)$ -arc. (This assertion is equivalent to the following one: no $[q + 1, N + 1, q - N + 1]_q$ generalized doubly-extended Reed-Solomon code can be extended to a $[q + 2, N + 1, q - N + 2]_q$ Maximal Distance Separable (MDS) code [3].) In [3], the following upper bound is obtained:

$$t(q) < \sqrt{q(3 \ln q + \ln \ln q + \ln 3)} + \sqrt{\frac{q}{3 \ln q}} + 4 \sim \sqrt{3q \ln q}.$$

The concept of *almost complete arc* in $\text{PG}(2, q)$ is considered in [15] where arcs of an infinite family $\mathcal{K}(q)$ are called almost complete if

$$(1) \quad \lim_{q \rightarrow \infty} \frac{\# \text{points not covered by } \mathcal{K}(q)}{\# \text{points of the plane } \text{PG}(2, q)} = 0.$$

An almost complete subset of a conic is an almost complete arc as the number of points not covered by it is smaller than q .

Almost complete arcs are useful for investigations of upper bounds on the smallest size of saturating sets and complete arcs in $\text{PG}(2, q)$.

In this work, we generalize both the aforementioned concepts.

Definition 2. (i) In $\text{PG}(3, q)$, an Almost Complete subset of the elliptic Quadric \mathcal{Q} (ACQ-subset, for short) is a proper subset of \mathcal{Q} covering all the points of $\text{PG}(3, q)$ except for the remaining points of \mathcal{Q} .

(ii) In $\text{PG}(N, q)$, $N \geq 2$, a cap \mathcal{K} is almost complete if the number of points not covered by \mathcal{K} is not greater than $\theta_{N-1, q}$.

Note that if caps of Definition 2(ii) form an infinite family of caps $\mathcal{K}(q)$ in the spaces $\text{PG}(N, q)$ with growing q then it holds that (cf. (1))

$$\lim_{q \rightarrow \infty} \frac{\# \text{points not covered by } \mathcal{K}(q)}{\# \text{points of the space } \text{PG}(2, q)} \leq \frac{\theta_{N-1, q}}{\theta_{N, q}} = 0.$$

An ACQ-subset is an almost complete cap as the number of points not covered by it is smaller than $q^2 + 1$.

Let $d(q)$ be the *smallest size of an ACQ-subset* in $\text{PG}(3, q)$.

Let $v(N, q)$ be the *smallest size of an almost complete cap* in $\text{PG}(N, q)$.

This work is devoted to *upper bounds* on $d(q)$ and $v(N, q)$.

The main results of this work are presented in

Theorem 1. (i) In $\text{PG}(3, q)$, for the smallest size of an ACQ-subset, we have

$$(2) \quad d(q) \leq (q+1)\sqrt{6\ln(q+1)} + 2q + 2 \sim q\sqrt{6\ln q}.$$

(ii) In $\text{PG}(N, q)$, for the smallest size of an almost complete cap, it holds that

$$(3) \quad v(N, q) \leq \sqrt{2N\theta_{N-1,q} \ln q} + 1 \sim q^{\frac{N-1}{2}} \sqrt{2N \ln q}, \quad N \geq 2.$$

Moreover, an almost complete cap of size at most $\sqrt{2N\theta_{N-1,q} \ln q} + 1$ can be constructed by a step-by-step greedy algorithm that in every step adds to the running cap a point providing the maximal possible (for the given step) number of new covered points.

One see that the bounds (2) and (3) asymptotically coincide with each other.

As far as it is known to the authors, ACQ-subsets and almost complete caps in $\text{PG}(N, q)$, $N \geq 3$, are not considered in the literature. Therefore, it remains for us only to compare the bounds (2) and (3) with the known bounds on the smallest size $t_2(N, q)$ of a complete cap in $\text{PG}(N, q)$. Of course, one should remember that these bounds are obtained for objects which are similar to the almost complete caps but not the same.

In [4], it is proved that

$$t_2(N, q) < cq^{\frac{N-1}{2}} \log^{300} q, \text{ with a constant } c \text{ independent of } q.$$

In [2], see also [6], under some probabilistic conjecture, it is shown that

$$(4) \quad t_2(N, q) < \frac{1}{q-1} \sqrt{q^{N+1}(N+1) \ln q} + \frac{\sqrt{q^{N+1}}}{q-3} \sim q^{\frac{N-1}{2}} \sqrt{(N+1) \ln q}.$$

We see that $q^{\frac{N-1}{2}} \sqrt{2N \ln q}$ is essentially smaller than $cq^{\frac{N-1}{2}} \log^{300} q$. On the other side, the bound $q^{\frac{N-1}{2}} \sqrt{2N \ln q}$ (that is *proved rigorously*) is greater than the conjectural bound (4). So, the bounds of Theorem 1, obtained in this work, seem to be reasonable.

These new concepts and the methods of their investigation can be useful for bounds and constructions of small saturating sets and small complete caps, including a rigorous proof of the conjectural bound (4).

This paper is organized as follows. In Section 2, the bound (2) is proved by probabilistic methods. In Section 3, the bound (3) is obtained by an algorithmic approach.

Some results of this work were briefly presented in [7].

2. An upper bound on the smallest size of an almost complete subset of an elliptic quadric in $\text{PG}(3, q)$

Let $w > 0$ be a fixed integer. Let \mathcal{Q} be an elliptic quadric in $\text{PG}(3, q)$. Consider a random $(w+1)$ -point subset $\mathcal{K}_{w+1} \subset \mathcal{Q}$. The total number of such subsets is $\binom{q^2+1}{w+1}$.

A fixed point A of $\text{PG}(3, q) \setminus \mathcal{Q}$ is covered by \mathcal{K}_{w+1} if it belongs to a bisecant of \mathcal{K}_{w+1} .

We denote by $\text{Prob}(\diamond)$ the probability of some event \diamond .

We estimate

$$\pi := \text{Prob}(A \text{ not covered by } \mathcal{K}_{w+1}),$$

as the ratio of the number of $(w+1)$ -point subsets of \mathcal{Q} not covering A over the total

number $\binom{q^2+1}{w+1}$ of subsets of \mathcal{Q} with size $(w+1)$. A set \mathcal{K}_{w+1} does not cover A if and only if every line through A contains at most one point of \mathcal{K}_{w+1} .

Through any point $A \in \text{PG}(3, q) \setminus \mathcal{Q}$, there are $\frac{q(q-1)}{2}$ bisecants and $q+1$ tangents of \mathcal{Q} [9]. Every bisecant has two places to put a point of \mathcal{K}_{w+1} while a tangent has the only one. For simplicity of presentation, we assume that a tangent also has two places to put a point of \mathcal{K}_{w+1} . (This will slightly worsen our estimates.) Therefore,

$$\pi < \frac{2^{w+1} \binom{q(q-1)/2 + q + 1}{w+1}}{\binom{q^2+1}{w+1}} = \frac{2^{w+1} \binom{(q^2+q+1)/2}{w+1}}{\binom{q^2+1}{w+1}},$$

where the numerator estimates from above the number of $(w+1)$ -point subsets of \mathcal{Q} not covering A . By straightforward calculations,

$$\begin{aligned} (5) \quad \pi &< \frac{(q^2+q+2)(q^2+q)(q^2+q-2)\dots(q^2+q+2-2i)\dots(q^2+q+2-2w)}{(q^2+1)(q^2)(q^2-1)\dots(q^2+1-i)\dots(q^2+1-w)} = \\ &= \prod_{i=0}^w \frac{q^2+q+2-2i}{q^2+1-i} = \prod_{i=0}^w \left(1 - \frac{i-1-q}{q^2+1-i}\right) < \prod_{i=0}^w \left(1 - \frac{i-1-q}{q^2+1}\right). \end{aligned}$$

From (5), using the inequality $1-x \leq e^{-x}$ for $x \neq 0$, we obtain that

$$\pi < e^{-\sum_{i=0}^w (i-1-q)/(q^2+1)} = e^{-(w^2-(2q+1)w-2q-2)/2(q^2+1)}.$$

Under the condition

$$(6) \quad w > \frac{4q^2+2q+2}{2q-1} = 2q+2 + \frac{4}{2q-1},$$

it holds that

$$-\frac{w^2-(2q+1)w-2q-2}{2(q^2+1)} < -\frac{(w-2q)^2}{2(q+1)^2},$$

whence

$$\pi < e^{-(w^2-(2q+1)w-2q-2)/2(q^2+1)} < e^{-(w-2q)^2/2(q+1)^2}.$$

The set \mathcal{K}_{w+1} is not ACQ-subset if at least one point of $\text{PG}(3, q) \setminus \mathcal{Q}$ is not covered by it. As $|\text{PG}(3, q) \setminus \mathcal{Q}| = q^3 + q$, we have

$$\begin{aligned} \text{Prob}(\mathcal{K}_{w+1} \text{ is not ASQ-subset}) &\leq \sum_{A \in \text{PG}(3, q) \setminus \mathcal{Q}} \text{Prob}(A \text{ not covered}) \leq \\ &\leq (q^3 + q)\pi < (q+1)^3 e^{-(w-2q)^2/2(q+1)^2}. \end{aligned}$$

The probability that all the points of $\text{PG}(3, q) \setminus \mathcal{Q}$ are covered by \mathcal{K}_{w+1} is

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is ACQ-subset}) > 1 - (q+1)^3 e^{-(w-2q)^2/2(q+1)^2}.$$

This probability is larger than 0 if one takes $w - 2q = \lceil (q+1)\sqrt{6\ln(q+1)} \rceil$,

where the condition (6) holds. This shows that there exists an ACQ-subset \mathcal{K}_{w+1} with size

$$w+1 \leq (q+1)\sqrt{6\ln(q+1)} + 2q + 2.$$

Theorem 1(i) is proved.

3. An upper bound on the smallest size of an almost complete cap in $\text{PG}(N, q)$

Assume that in $\text{PG}(N, q)$, $N \geq 2$, a cap is constructed by a step-by-step greedy algorithm (*Algorithm*, for short) which in every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points. Such approach is considered in [1, 2, 6].

After the w -th step of *Algorithm*, a w -cap \mathcal{K}_w is obtained that does not cover exactly U_w points.

Denote by $\mathcal{U}(\mathcal{K})$ the set of points of $\text{PG}(N, q)$ that are not covered by a cap \mathcal{K} . By above, $\#\mathcal{U}(\mathcal{K}_w) = U_w$. Let the cap \mathcal{K}_w consist of w points A_1, A_2, \dots, A_w . Let $A_{w+1} \in \mathcal{U}(\mathcal{K}_w)$ be the point that will be included into the cap in the $(w+1)$ -st step.

A point A_{w+1} defines a bundle $\mathcal{B}(A_{w+1})$ of w unisecants to \mathcal{K}_w which are denoted as $\overline{A_1 A_{w+1}}, \overline{A_2 A_{w+1}}, \dots, \overline{A_w A_{w+1}}$, where $\overline{A_i A_{w+1}}$ is the unisecant connecting A_{w+1} with the cap point A_i . Every unisecant contains $q+1$ points. Except for A_1, \dots, A_w , all the points on the unisecants in the bundle are candidates to be new covered points in the $(w+1)$ -st step. We call $\{A_{w+1}\}$ and $\mathcal{B}(A_{w+1}) \setminus (\mathcal{K}_w \cup \{A_{w+1}\})$, respectively, the *head* and the *basic part* of the bundle $\mathcal{B}(A_{w+1})$. For a given cap \mathcal{K}_w , in total, there are $\#\mathcal{U}(\mathcal{K}_w) = U_w$ distinct bundles.

Let $\Delta_w(A_{w+1})$ be the number of new covered points in the $(w+1)$ -st step, i.e.,

$$(7) \quad \Delta_w(A_{w+1}) = \#\mathcal{U}(\mathcal{K}_w) - \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}).$$

In future, we consider continuous approximations of the discrete functions $\Delta_w(A_{w+1})$, $\#\mathcal{U}(\mathcal{K}_w)$, $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\})$, keeping the same notations.

We take into account that *all points that are not covered by a cap lie on unisecants to the cap*.

In total there are $\theta_{N-1,q}$ lines through every point of $\text{PG}(N, q)$. Therefore, through every point A_i of \mathcal{K}_w , there is a pencil $\mathcal{P}(A_i)$ of $\theta_{N-1,q} - (w - 1)$ unisecants to \mathcal{K}_w , where $i = 1, 2, \dots, w$. The total number T_w^Σ of the unisecants to \mathcal{K}_w is

$$(8) \quad T_w^\Sigma = w(\theta_{N-1,q} + 1 - w).$$

Let $\gamma_{w,j}$ be the number of uncovered points on the j -th unisecant \mathcal{T}_j , $j = 1, 2, \dots, T_w^\Sigma$.

Every uncovered point lies on exactly w unisecants; due to this *multiplicity*, on all unisecants there are in total Γ_w^Σ uncovered points, where

$$(9) \quad \Gamma_w^\Sigma = \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w.$$

By (8), (9), the average number γ_w^{aver} of uncovered points on a unisecant is

$$(10) \quad \gamma_w^{\text{aver}} = \frac{\Gamma_w^\Sigma}{T_w^\Sigma} = \frac{U_w}{\theta_{N-1,q} + 1 - w}.$$

A unisecant \mathcal{T}_j belongs to $\gamma_{w,j}$ distinct bundles, as every uncovered point on \mathcal{T}_j may be the head of a bundle. Moreover, \mathcal{T}_j provides $\gamma_{w,j}(\gamma_{w,j} - 1)$ uncovered points to the basic parts of all these bundles. The noted points are counted with *multiplicity*.

Taking into account the *multiplicity*, in all U_w the bundles there are

$$(11) \quad \sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}(\gamma_{w,j} - 1),$$

uncovered points, where U_w is the total numbers of all the heads. By (9), (11),

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 - \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = U_w(1 - w) + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2.$$

For a cap \mathcal{K}_w , we denote by $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ the average value of $\Delta_w(A_{w+1})$ by all $\#\mathcal{U}(\mathcal{K}_w)$ uncovered points A_{w+1} , i.e.,

$$(12) \quad \Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{\#\mathcal{U}(\mathcal{K}_w)} = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{U_w} = \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w} - w + 1 \geq 1,$$

where the inequality is obvious by sense; also note that

$$(13) \quad \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \geq \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w.$$

We denote a lower estimate of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$, see Lemma 1 below, as follows:

$$(14) \quad \Delta_w^{\text{rigor}}(\mathcal{K}_w) := \max \left\{ 1, \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1 \right\} = \begin{cases} \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1 & \text{if } U_w \geq \theta_{N-1,q} + 1 - w, \\ 1 & \text{if } U_w < \theta_{N-1,q} + 1 - w. \end{cases}$$

Lemma 1. For a w -cap \mathcal{K}_w , the following holds:

- This inequality always holds

$$(15) \quad \Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w).$$

- In (15), we have the equality

$$(16) \quad \Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1,$$

if and only if every unisecant contains the same number $\frac{U_w}{\theta_{N-1,q} + 1 - w}$ of uncovered points where $\frac{U_w}{\theta_{N-1,q} + 1 - w}$ is integer.

- In (15), the equality

$$(17) \quad \Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = 1,$$

holds if and only if each unisecant contains at most one uncovered point.

Proof. By Cauchy-Schwarz-Bunyakovsky inequality, it holds that

$$\left(\sum_{j=1}^T \gamma_{w,j} \right)^2 \leq T_w \sum_{j=1}^T \gamma_{w,j}^2,$$

where equality holds if and only if all $\gamma_{w,j}$ coincide. In this case $\gamma_{w,j} = \frac{U_w}{\theta_{N-1,q} + 1 - w}$

for all j and, moreover, the ratio $\frac{U_w}{\theta_{N-1,q} + 1 - w}$ is integer. Now, by (8), (9), we have

$$\frac{wU_w}{\theta_{N-1,q} + 1 - w} \leq \frac{\sum_{j=1}^T \gamma_{w,j}^2}{U_w},$$

that together with (9), (12), (13), (14) gives (15)-(17). \square

Remark 1. One can treat the estimates (15), (16) as follows. A bundle contains w unisecants having a common point, its head. Therefore the average number of uncovered points in a bundle is $w\gamma_w^{\text{aver}} - (w - 1)$ where γ_w^{aver} is defined in (10) and the term $w - 1$ takes into account the common point.

By (7) and Lemma 1,

$$U_{w+1} \leq U_w \left(1 - \frac{w}{\theta_{N-1,q} + 1 - w} \right) + w - 1 < U_w \left(1 - \frac{w}{\theta_{N-1,q}} \right) + w$$

whence

$$(18) \quad \begin{aligned} U_{w+1} - \theta_{N-1,q} &< U_w \left(1 - \frac{w}{\theta_{N-1,q}} \right) + w - \theta_{N-1,q} = \\ &= U_w \left(\frac{\theta_{N-1,q} - w}{\theta_{N-1,q}} \right) - (\theta_{N-1,q} - w) = \left(1 - \frac{w}{\theta_{N-1,q}} \right) (U_w - \theta_{N-1,q}). \end{aligned}$$

By (18)

$$\begin{aligned}
(19) \quad U_2 - \theta_{N-1,q} &< \left(1 - \frac{1}{\theta_{N-1,q}}\right) (U_1 - \theta_{N-1,q}); \\
U_3 - \theta_{N-1,q} &< \left(1 - \frac{2}{\theta_{N-1,q}}\right) (U_2 - \theta_{N-1,q}) = \\
&= \left(1 - \frac{2}{\theta_{N-1,q}}\right) \left(1 - \frac{1}{\theta_{N-1,q}}\right) (U_1 - \theta_{N-1,q}); \\
&\quad \dots \\
U_{w+1} - \theta_{N-1,q} &< \left(1 - \frac{w}{\theta_{N-1,q}}\right) \dots \left(1 - \frac{2}{\theta_{N-1,q}}\right) \left(1 - \frac{1}{\theta_{N-1,q}}\right) (U_1 - \theta_{N-1,q}) = \\
&= (U_1 - \theta_{N-1,q}) f_q(w),
\end{aligned}$$

where

$$f_q(w) = \prod_{i=1}^w \left(1 - \frac{i}{\theta_{N-1,q}}\right).$$

Remark 2. The function $f_q(w)$ and its approximations, including (21), appear in distinct tasks of Probability Theory, e.g., in the Birthday problem (or the Birthday paradox) [5, 13]. Actually, let the year contain $\theta_{N-1,q}$ days and let all birthdays occur with the same probability. Then $P_{\theta_{N-1,q}}^\#(w+1) = f_q(w)$, where $P_{\theta_{N-1,q}}^\#(w+1)$ is the probability that no two persons from $w+1$ random persons have the same birthday. Moreover, if birthdays occur with different probabilities we have $P_{\theta_{N-1,q}}^\#(w+1) < f_q(w)$ [5].

By (19), taking into account that $U_1 = \theta_{N,q} - 1 < \theta_{N,q} = \theta_{N-1,q} + q^N$, we have

$$(20) \quad U_{w+1} < q^N f_q(w) + \theta_{N-1,q}.$$

Using the inequality $1 - x \leq e^{-x}$ for $x \neq 0$, we obtain

$$(21) \quad f_q(w) < \prod_{i=1}^w e^{-i/\theta_{N-1,q}} = e^{-(w^2+w)/2\theta_{N-1,q}} < e^{-w^2/2\theta_{N-1,q}}.$$

Let

$$(22) \quad w = \left\lceil \sqrt{2\theta_{N-1,q} \ln q^N} \right\rceil = \left\lceil \sqrt{2N\theta_{N-1,q} \ln q} \right\rceil \sim q^{\frac{N-1}{2}} \sqrt{2N \ln q}.$$

Then, by (20)-(22),

$$w^2 = 2\theta_{N-1,q} \ln q^N;$$

$$e^{-w^2/2\theta_{N-1,q}} = \frac{1}{q^N};$$

$$U_{w+1} < \theta_{N-1,q} + 1;$$

$$U_{w+1} \leq \theta_{N-1,q}.$$

So, the number of points of $\text{PG}(N, q)$ not covered by the cap \mathcal{K}_{w+1} is at most $\theta_{N-1,q}$.

We have proved Theorem 1(ii).

Acknowledgements: The research of A. A. Davydov was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project 14-50-00150). The research of S. Marcugini and F. Pambianco was supported in part by Ministry for Education, University and Research of Italy (MIUR) (Project “Geometrie di Galois e strutture di incidenza”) and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA-INDAM).

References

1. Bartoli, D., A. A. Davydov, G. Faina, A. A. Kreshchuk, S. Marcugini, F. Pambianco. Upper Bounds on the Smallest Size of a Complete Arc in $PG(2, q)$ under a Certain Probabilistic Conjecture. – Problems Inform. Transm., Vol. **50**, 2014, No 4, pp. 320-339.
2. Bartoli, D., A. A. Davydov, G. Faina, S. Marcugini, F. Pambianco. Conjectural Upper Bounds on the Smallest Size of a Complete ap in $PG(N, q)$, $N \geq 3$. – Electron. Notes Discrete Math., Vol. **57**, 2017, pp. 15-20.
3. Bartoli, D., A. A. Davydov, S. Marcugini, F. Pambianco. On the Smallest Size of an Almost Complete Subset of a Conic in $PG(2, q)$ and Extendability of Reed-Solomon Codes. – arXiv:1609.05657 [math.CO], 2016.
4. Bartoli, D., G. Faina, S. Marcugini, F. Pambianco. A Construction of Small Complete Caps in Projective Spaces. – J. Geom., Vol. **108**, 2017, No 1, pp. 215-246.
5. Clevenston, M. L., W. Watkins. Majorization and the Birthday Inequality. – Math. Magazine, Vol. **64**, 1991, No 3, pp. 183-188.
6. Davydov, A. A., G. Faina, S. Marcugini, F. Pambianco. Upper Bounds on the Smallest Size of a Complete Cap in $PG(N, q)$, $N \geq 3$, under a Certain Probabilistic Conjecture. – arXiv:1706.01941 [math.CO], 2017.
7. Davydov, A. A., S. Marcugini, F. Pambianco. Upper Bounds on the Smallest Size of an Almost Complete Cap in $PG(N, q)$. – In: Proc. of 8th International Workshop on Optimal Codes and Related Topics, OC’17 (in Second International Conference “Mathematics Days in Sofia”), Sofia, Bulgaria, 2017, pp. 67-72.
8. Hirschfeld, J. W. P. Projective Geometries over Finite Fields. Second Edition. Oxford, Oxford University Press, 1998.
9. Hirschfeld, J. W. P. Finite Projective Spaces of Three Dimensions. Oxford, Oxford University Press, 1985.
10. Hirschfeld, J. W. P., L. Storme. The Packing Problem in Statistics, Coding Theory and Finite Geometry: Update 2001. – In: A. Blokhuis, J. W. P. Hirschfeld et al. Eds. Proc. of 4th Isle of Thorns Conf., Chelwood Gate, 2000, Kluwer Academic Publisher, Boston, Finite Geometries, Developments of Mathematics, Vol. **3**, 2001, pp. 201-246.
11. Hirschfeld, J. W. P., J. A. Thas. Open Problems in Finite Projective Spaces. – Finite Fields and their Appl., Vol. **32**, 2015, No 1, pp. 44-81.
12. Kovács, S. J. Small Saturated Sets in Finite Projective Planes. – Rend. Mat. (Roma), Vol. **12**, 1992, No 1, pp. 157-164.
13. Sayrafiezadeh, M. The Birthday Problem Revisited – Math. Magazine, Vol. **67**, 1994, No 3, pp. 220-223.
14. Storme, L. Completeness of Normal Rational Curves. – J. Algebraic Combin., Vol. **1**, 1992, No 2, pp. 197-202.
15. Ughi, E. Small Almost Complete Arcs. – Discrete Math., Vol. **255**, 2002, No 3, pp. 367-379.

Received 25.09.2017; Accepted 01.12.2017