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Research Article

Heat rectification in He II counterflow in radial geometries

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Abstract

We consider heat rectification in radial flows of turbulent helium II, where heat flux is not described by Fourier's law, but by a more general law. This is different from previous analyses of heat rectification, based on such law. In our simplified analysis we show that the coupling between heat flux and the gradient of vortex line density plays a decisive role in such rectification. Such rectification will be low at low and high values of the heat rate, but it may exhibit a very high value at an intermediate value of the heat rate. In particular, for a given range of values for the incoming heat flow, the outgoing heat flow corresponding to the exchange of internal and external temperatures would be very small. This would imply difficulties in heat removal in a given range of temperature gradients.

Keywords: `Heat rectification, Superfluid helium, Radial heat flux, Quantum turbulence`

AMS subject classification: `82D50, 80A20, 76Fxx`

1. Introduction

Heat rectification has become a very active topic of research in the so-called phononics [1–4]. The phenomenon of rectification consists of a difference in the value of the heat flux when a same temperature gradient, but in opposite directions, is imposed on a system. This phenomenon is observed in inhomogeneous systems (inhomogeneities of composition or of geometry), and it is characterized by the value of the rectification coefficient $R \equiv q_R/q_D$ (with q_D and q_R the respective absolute values of the heat flux in two opposite directions, usually called direct and reverse directions). Alternatively, it may manifest itself as a different temperature gradient for the same absolute value of the heat flux for opposite directions of the flux. Thermal rectification has been studied in many systems [5–16], but we are not aware of similar analyses in superfluid helium (He II), which is known to have a very peculiar behaviour in heat transfer.

Helium II has a very interesting behaviour, due to its quantum nature. The most remarkable aspects are its vanishingly small viscosity, and its extremely high thermal conductivity, several order of magnitude larger than high-conductivity liquids or solids. Further, He II is unable to boil and temperature waves can propagate in it. One of the most typical effects in liquid helium II is the so-called *counterflow superfluid turbulence*, whose physical picture is a tangle of vortices of equal circulation $\kappa = h/m$, with h the Planck constant and m the mass of helium atom [17]. This kind of turbulence is generated thermally, applying a heat flux, exceeding a critical value q_c [18–20].

In usual materials and for a given value of heat flux, the temperature distribution is obtained by integrating Fourier's law

$$q(z) = -\lambda[T(z), X(z)]\frac{dT}{dz} \tag{1}$$

where $q(z)$ is the heat flux, λ the thermal conductivity, T the temperature, and X may be a composition of geometry parameters which changes along the length of the system, whose position along it is described by the value of z. In one–dimensional problems and in steady state $\dot{Q}(z)$ is constant, with $\dot{Q}(z)$ being the total heat flow across the system at position z, namely $\dot{Q}(z) = q(z)A(z)$, with $A(z)$ the transverse area at position z. If λ depends only on T, no rectification is found, but if it depends also on some other variable, rectification may be achieved in principle.

The interest of exploring heat rectification in superfluid He II goes beyond a relatively trivial extension of previous works because, in difference to solids, the behaviour of heat transport in He II is much richer and complex. In particular, heat flux does not follow Fourier's law (1), but more complicated expressions which will be commented in Section 2. Further, in this quantum fluid a new type of turbulence takes place, the so-called counterflow superfluid turbulence, that is due to the formation of quantized vortex lines in the superfluid. The simplest way to describe the vortex tangle is in terms of a single scalar quantity L, the total length of vortex lines per unit volume (*vortex line density*, for short). There are different kinds of turbulence, depending on the intensity of the heat flux and on the form and dimension of the channel. As illustrated in [21] the presence of a superfluid vortex tangle modifies the effective heat conductivity of this fluid. This suggests to explore the possibility of rectification in heat transport in tubes filled with turbulent superfluid helium.

A new aspect of our analysis is the cylindrical geometry with radial heat flux, in contrast to usual analyses related to heat flow along a channel. After presenting in Section 2 the starting equation for our analysis, describing the heat flux in He II, which was derived in previous papers, in Section 3 we apply it for the first time to the study of heat rectification. In the concluding remarks we discuss the several contributions to the rectification.

2. Heat transport in laminar and turbulent He II.

Heat transport in superfluid He II is not described by Fourier's law but it requires a much more general law, considering the long relaxation time τ_1 of the heat flux $\mathbf{q}$ (first term of Equation (2)), nonlocal effects (fourth term of Equation (2)) related to the long coherent length of the superfluid (a macroscopic quantum coherence), and nonlinear effects (third term and last term of Equation (2)) related to quantized vortices forming a turbulent tangle of lines described by the vortex length per unit volume L [22–24]. Such generalized equation takes the form [24,25]:

$$\frac{1}{\zeta}\frac{d\mathbf{q}}{dt} + \nabla T + \frac{\chi}{\zeta}\nabla L - \tilde{\beta}^2 T^3 \nabla^2 \mathbf{q} = -\frac{1}{\lambda_1}\mathbf{q} - \frac{KL}{\zeta}\mathbf{q} \tag{2}$$

where $\tilde{\beta} = -1/(ST^2)$, with S the entropy per unit volume and T the absolute temperature, $\zeta = \lambda_1/\tau_1$, with λ_1 the intrinsic thermal conductivity of the superfluid, and τ_1 the relaxation time of the heat flux, χ a coupling parameter between $\mathbf{q}$ and ∇L, and K the mutual friction coefficient between the vortex lines and the heat carriers, mainly phonons and rotons, through their corresponding collisions. Equations like (2) may be analyzed in general by using extended thermodynamics [26–28] and have been applied to the description of heat transport in several concrete situations [21,29,30]. A detailed discussion on the sign of coefficient χ was made in [22]. We have seen that both thermodynamical considerations and diffusion effects lead to $\chi < 0$. Note that Equation (2) would reduce to Fourier's law (1) if the terms in L, in τ_1 and in β disappear.

The coefficients λ_1 and τ_1 are very high, but the ratio $\zeta = \lambda_1/\tau_1$ is finite, and it is related to the speed of the second sound through $V_2^2 = \frac{\zeta}{\rho c_V}$ (of the order of 20 m/s at 1.5 K [18]), with ρ the mass density and c_V the specific heat per unit mass [26,27]. L is the vortex line density of the quantized vortex tangle which is formed when the heat flux exceeds some critical values, and it must be described by means of its own evolution equation. The term $\mathbf{q}/\lambda_1$ in Equation (2) may usually be neglected as compared to $(\beta^2 T^3 \nabla^2 \mathbf{q})$ and to $(KL/\zeta)\mathbf{q}$. Indeed, in a cylindrical channel of radius R the first of these two latter terms is of the order of $(S^2TR^2)^{-1}\mathbf{q}$; the inequality $\lambda_1 \gg S^2TR^2$ is widely satisfied at low T and small R; furthermore, referring to the friction term, $\lambda_1 \gg \zeta/(KL)$ for sufficiently high values of L.

For inhomogeneous systems we have for the evolution equation of L [19,20,24].

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha_0 V_{ns} - \omega' \frac{\beta\kappa}{d}\right] L^{3/2} + D\nabla^2 L \tag{3}$$

where $\kappa = h/m$ is the quantum of circulation ($9.9 * 10^{-8}$ m^2/s), β, α_0 and ω' parameters depending on T, with β (of the order of 0.16 at 1.5 K [31]) and α_0 (of the order of $3.2 * 10^{-8}$ at 1.5 K [32]) describing the rate of vortex destruction and vortex generation per unit volume, and ω' describing the influence of the walls on the vortex generation and d the channel diameter. D is the vortex diffusion coefficient describing the flow of vortex lines from zones of higher L to zones of lower L, and whose value is of the order of κ (times a numerical factor between 1 and 5), depending on temperature, but which is not yet known in detail. According to [33] $D = 2.2\kappa$ cm^2/s at $T = 1.5K$. In Equation (3), wall effects are simply described by the term in κ/d in the right hand side. However, more detailed recent analyses indicate the rich and complex behaviour of the superfluid in boundary layers close to the walls [34,35].

$V_{ns} = | < \mathbf{v}_n - \mathbf{v}_s > | = |\mathbf{q}|/(\rho_s sT)$ is the so-called counterflow velocity (here $|\mathbf{q}|$ is the modulus of the heat flux), and $\mathbf{v}_n$ and $\mathbf{v}_s$ are normal fluid and superfluid velocity, ρ_s is the density of the superfluid component and s the specific entropy [24]. For $\omega' = 0$ and $D = 0$ Equation (3) reduces to the usual Vinen equation [19,20,36], which describes fully developed turbulence in homogeneous wide channels.

For small enough V_{ns} the flow is laminar ($L = 0$). For quantum Reynolds number $Rey \equiv \frac{V_{ns}d}{\kappa}$ such that $Rey \geq \frac{\alpha_0}{\omega'\beta}$ in (3), the $L = 0$ solution becomes unstable and turbulence TI appears (a mild form of turbulence with relatively low vortex line density). This happens for $Rey \equiv \frac{V_{ns}d}{\kappa} \geq 127$ ($T = 1.5$ K) or 96 ($T = 1.7$ K). For $Rey \equiv \frac{V_{ns}d}{\kappa} \geq 219$ ($T = 1.5$ K) or 186 ($T = 1.7$ K), appears TII turbulence, namely a stronger form of turbulence, with considerably higher values of L. The stable solution of (3) becomes:

$$L^{1/2} d = \frac{\alpha_0}{\beta} Rey - \omega' \tag{4}$$

thus yielding $L \neq 0$, and increasing with $Rey = \frac{|\mathbf{q}|d}{(\rho_s sT)\kappa}$. Note that expression (4) for L is valid both in the TI and in the TII regimes, but with different values of α_0 and ω' [21,22]. In our analysis in Section 3 we will assume high values of Rey, so that ω' will be neglected. In wide channels may yet appear an additional form of turbulence, dubbed as TIII turbulence, in high–aspect ratio channels with an accumulation of vortices, which cannot be fully grasped by the present simple approximation.

When these results are introduced in (2), one obtains equations relating $\mathbf{q}$ to ∇T, ∇L and L, which are strongly nonlinear and considerably different from Fourier's law.

3. Heat rectification in radial turbulent flow.

Normally, heat flow in He II is considered in longitudinal tubes with constant cross section. However, we have also focused a part of our analyses in inhomogeneous situations where L changes with position [37]. An especially, relevant and simple situation is that of cylindrical radial flows [29,30,37], which is relevant, for instance, in the cooling of cylindrical systems through superfluid helium.

In a previous paper on inhomogeneous vortex tangles [30] we considered a radial counterflow from a cylindrical wall of radius R_0 at temperature T_0 to another concentric cylinder of radius $R_1 > R_0$ at temperature $T_1 < T_0$, or viceversa. In this situation, the behavior of L is no longer homogeneous because it depends on the radial distance r to the central axis. The origin of inhomogeneity of vortex line density is the inhomogeneity of the heat flux itself according to (4). Because of the geometry of the flow, the heat flux is maximum near the center and decreases towards the external wall, as the total heat flow $\dot{Q}$ is constant. The source of vortices is everywhere (corresponding to the source term of the local Vinen's equation), but more intense in the inner region. In particular, in [30] we found the possibility of hysteresis, namely, a different behaviour of $\mathbf{q}$ for increasing and decreasing (in time) radial temperature gradients. Here we consider steady–state situations and compare the behaviour of the system for outwards and inwards temperature gradients (or heat flows), instead of the behaviour for time increasing and decreasing temperature gradients.

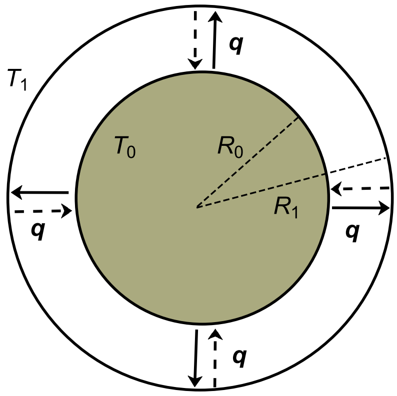

Figure 1. The heat flux $\mathbf{q}$ can flow from the inner cylinder at temperature T_0 to the outer cylinder at lower temperature T_1, or viceversa, if the temperatures T_0 and T_1 of the inner and outer cylinders are mutually exchanged. Heat rectification means that the corresponding inwards and outwards heat flows have different absolute values.

In the steady case the heat flux goes from one to the other cylinder as the inverse of the radius r, namely $q_r \equiv \frac{\dot{Q}}{2\pi r}$, with $\dot{Q}$ being the total heat flowing radially per unit time and unit length of the cylinder; then the vortex line density also depends on r, because the source of vortices is related to the heat flux. Note that here we have assumed $\mathbf{q} = q_r\hat{\mathbf{r}}$. The form $q_r \sim r^{-1}$ comes from the steady–state condition $\nabla \cdot \mathbf{q} = 0$, applied to cylindrical geometry.

If we consider the heat flux going from the inner cylinder to the outer we have $\dot{Q} > 0$, $q_r > 0$ and $\frac{dq_r}{dr} < 0$, (viceversa if the heat flux goes from the external cylinder to the inner we have $\dot{Q} < 0$, $q_r < 0$ and $\frac{d|q_r|}{dr} < 0$).

The heat transport Equation (2) in steady-state after neglecting the term $\mathbf{q}/\lambda_1$, and recalling that χ is a negative coefficient [22], reduces to:

$$\frac{\partial T}{\partial r} = \frac{|\chi|}{\zeta}\frac{\partial L}{\partial r} - \frac{K}{\zeta}L\frac{\dot{Q}}{2\pi r} + \frac{l^2}{r}\frac{\partial}{\partial r}\left[r\frac{\partial q_r}{\partial r}\right] \tag{5}$$

with $l^2 \equiv \tilde{\beta}^2 T^3$.

Neglecting ω' in (4) (or, equivalently, assuming that Rey is high enough, as in well developed TII turbulent regime), the expression for the vortex line density is $L = \gamma'^2\frac{\dot{Q}^2}{r^2}$, with $\gamma' = \alpha_0/(4\pi^2\beta)$. Then we obtain

$$\frac{\partial T}{\partial r} = -\frac{1}{r^3}\left[2\frac{|\chi|}{\zeta}\gamma'^2\dot{Q}^2 + \frac{K}{\zeta}\gamma'^2\frac{\dot{Q}^3}{2\pi} - l^2\frac{\dot{Q}}{2\pi}\right]. \tag{6}$$

If one neglects for simplicity the temperature dependence of χ, ζ, γ' and K, and one considers a steady–state situation with $\dot{Q}$ constant in time, Equation (6) may be written as $\frac{\partial T}{\partial r} = -\frac{\alpha}{r^3}$, with α the corresponding quantity in parentheses in the right–hand side of (6). By integrating (6) we obtain:

$$T(R_1) - T(R_0) = \frac{\alpha}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_0^2}\right]. \tag{7}$$

If we consider the same total absolute value of heat flux $|\dot{Q}|$ but in opposite directions we obtain different values of the temperature gradient, i. e. for the difference $T(R_1) - T(R_0)$.

An interesting consequence of this kind of flow is heat rectification, i. e. the value $\dot{Q}_{out}$ of the outwards heat flow for $T_0 = T(R_0)$, $T_1 = T(R_1)$ may be compared with the value $\dot{Q}_{in}$ when the temperatures T_0, T_1 are applied to the outer and the inner cylinders, respectively.

- For $\dot{Q} > 0$ (outgoing flux) we have:

$$T(R_1) - T(R_0) = \frac{\alpha_{out}}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_0^2}\right] \tag{8}$$

with $T(R_1) < T(R_0)$, and
$\alpha_{out} = \frac{K}{\zeta}\frac{\gamma'^2}{2\pi}|\dot{Q}_{out}|^3 + 2\frac{|\chi|}{\zeta}\gamma'^2|\dot{Q}_{out}|^2 - \frac{l^2}{2\pi}|\dot{Q}_{out}|$.

- For $\dot{Q} < 0$ (ingoing flux) we have:

$$T(R_1) - T(R_0) = \frac{\alpha_{in}}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_0^2}\right] \tag{9}$$

with $T(R_1) > T(R_0)$, and
$\alpha_{in} = -\frac{K}{\zeta}\frac{\gamma'^2}{2\pi}|\dot{Q}_{in}|^3 + 2\frac{|\chi|}{\zeta}\gamma'^2|\dot{Q}_{in}|^2 + \frac{l^2}{2\pi}|\dot{Q}_{in}|$

Note that the signs of the terms in odd powers of $|\dot{Q}|$ have been changed, in difference to the sign of the term in $|\dot{Q}|^2$.

For a same absolute value of $\dot{Q}$, i.e. for $|\dot{Q}_{out}| = |\dot{Q}_{in}| = |\dot{Q}|$, the effective temperature gradients $(T(R_1) - T(R_0))/(R_1 - R_0)$ in the outward direction and in the inner direction are related as:

$$\frac{(T(R_1) - T(R_0))_{out}/(R_1 - R_0)}{(T(R_1) - T(R_0))_{in}/(R_1 - R_0)} = -\frac{K|\dot{Q}|^2\gamma'^2 + 4\pi|\chi|\gamma'^2|\dot{Q}| - l^2\zeta}{K|\dot{Q}|^2\gamma'^2 - 4\pi|\chi|\gamma'^2|\dot{Q}| - l^2\zeta}. \tag{10}$$

This result is obtained by taking the ratio of (8) and (9), and taking into account that the ratio of $\frac{(\partial T/\partial R)_{out}}{(\partial T/\partial R)_{in}}$ is the same as the ratio of $\frac{(\partial T/\partial(1/R^2))_{out}}{(\partial T/\partial(1/R^2))_{in}}$, and considering small the difference $(R_1 - R_0)$.

Note that when $\chi = 0$, the second member of (10) becomes -1, which corresponds indeed to the mutual exchange of the values of $T(R_1)$ and $T(R_0)$ in the inner and outward situations.

According to (10), the absolute value of the quantity $\frac{(\partial T/\partial R)_{out}}{(\partial T/\partial R)_{in}}$ is close to 1 when $\dot{Q}$ is zero or ∞, instead it has very high values (in fact, it diverges) when the denominator is zero. Then, it is seen that rectification disappears for very high or very small values of $|\dot{Q}|$, but that it may be very high for $|\dot{Q}|$ close to:

$$|\dot{Q}^*| = \frac{2\pi|\chi|}{K}\left[\sqrt{1 + \frac{l^2\zeta K}{4\pi^2\chi^2\gamma'^2}} + 1\right] \tag{11}$$

For high values of $|\dot{Q}|$ expression (10) may be approximated as:

$$\frac{(\partial T/\partial R)_{out}}{(\partial T/\partial R)_{in}} = -\left[1 + \frac{8\pi|\chi|}{K}\frac{1}{|\dot{Q}|}\right]. \tag{12}$$

The limit of small values of $|\dot{Q}|$ may also be mathematically considered, but if $|\dot{Q}|$ is too small the system will be in laminar state, without quantized vortices. Since here we are paying special attention to the influence of vortices, we will not consider this limit.

It would also be possible to obtain the more usual rectification coefficient $R \equiv |\dot{Q}_{out}|/|\dot{Q}_{in}|$ (note that this definition is more suitable than q_R/q_D for inhomogeneous flows) by imposing a same temperature gradient in modulus $|\partial T/\partial r|$ but going outwards and inwards.

In this way, if we impose that $\frac{(T(R_1)-T(R_0))_{out}/(R_1-R_0)}{(T(R_1)-T(R_0))_{in}/(R_1-R_0)} = -1$, i.e., from (8) and (9), $\alpha_{out} = -\alpha_{in}$, we get

$$\begin{aligned} &\frac{K}{\zeta}\frac{\gamma'^2}{2\pi}|\dot{Q}_{out}|^3 + 2\frac{|\chi|}{\zeta}\gamma'^2|\dot{Q}_{out}|^2 - \frac{l^2}{2\pi}|\dot{Q}_{out}| = \\ &\qquad \frac{K}{\zeta}\frac{\gamma'^2}{2\pi}|\dot{Q}_{in}|^3 - 2\frac{|\chi|}{\zeta}\gamma'^2|\dot{Q}_{in}|^2 - \frac{l^2}{2\pi}|\dot{Q}_{in}|. \end{aligned} \tag{13}$$

For a given $|\dot{Q}_{out}|$ this equation yields the corresponding $|\dot{Q}_{in}|$ and the rectification coefficient R may be found.

Then, dividing (13) by $|\dot{Q}_{in}|$, we get for R the following equation

$$|\dot{Q}_{in}|^2 K\gamma'^2 R^3 - 4\pi|\chi|\gamma'^2|\dot{Q}_{in}|R^2 - l^2\zeta R = |\dot{Q}_{in}|^2 K\gamma'^2 + 4\pi|\chi|\gamma'^2|\dot{Q}_{in}| - l^2\zeta. \tag{14}$$

For a given $|\dot{Q}_{in}|$ (14) yields the value of the rectification coefficient R. If $\chi = 0$, one obtains from (14) $R = 1$, and there is no rectification, as already mentioned in the paragraphs below Equation (10).

For the value of $|\dot{Q}_{in}|$ for which the right-hand side of (14) is 0 we have the solution $R = 0$ and

$$R = \frac{\sqrt{1 + \frac{l^2\zeta K}{4\pi^2\chi^2\gamma'^2}} - 1}{\sqrt{1 + \frac{l^2\zeta K}{4\pi^2\chi^2\gamma'^2}} + 1} \tag{15}$$

which tend to 1 (no rectification) when $\chi \to 0$. There is also a negative solution without physical meaning. The values of R close to zero indicate a very high rectification, because exchanging the internal and external temperatures, the value of the heat flux is very different. In particular, $|\dot{Q}_{out}|$ becomes much smaller that the corresponding $|\dot{Q}_{in}|$. However, for too small values of the heat flux the system will not be turbulent, and the contributions in L should be cancelled, thus modifying Equation (5). This makes that in our analysis in the present paper, the solution $R = 0$ should not be taken into account as a truly physical solution, because it does not correspond to the hypothesis of a turbulent system, so that solution (15) is the physical solution.

4. Concluding remarks.

Our analysis of heat rectification in Section 3 has started from an equation for the heat flux more general than Fourier's equation, namely Equation (2); this is an original aspect of this paper, because previous analyses of heat rectification in the literature always start from Fourier's equation.

Our analysis has shown that the rectification depends on the coefficient χ coupling the heat flux to the gradient in L, as seen in Equation (10); note however that our analysis has been simplified in that we have neglected the temperature dependence of the coefficients. However, despite this simplification (that will be removed in future analyses), this has been sufficient to show rectification in this situation, thus pointing a proof of concept of interest of this system.

In expression (10) it has been seen that rectification arises in this case because of the coupling between the heat flux and the gradient of L. Indeed the terms in $\mathbf{q}$ and $L\mathbf{q}$ change sign when $\mathbf{q}$ is reversed but since L depends on $\mathbf{q}^2$, ∇L does not change sign under such reversal. For an homogeneous system, with L constant, one would not have rectification because such term would be zero, but in our radial case L changes with the position. Therefore, this situation may be compared to that arising in so-called "graded systems" in solid systems [38]. Such systems are characterized by a composition depending on the position, for instance Ge_xSi_{1-x}, with the stoichiometric index x changing with position z along the system. In our case, what is changing with the radial position is the value of the vortex line density L. This analogy should not make us forget that the starting equation for heat transport in our case was not Fourier's equation, but a more general one.

A particularly interesting result is that the rectification ratio (10) may be very high close to value of $|\dot{Q}|$ given by (11). Analogously, the usual rectification coefficient R becomes close to zero for values of $|\dot{Q}_{in}|$ close to the value given by (11). In this case $|\dot{Q}_{out}|$ becomes much smaller than $|\dot{Q}_{in}|$. This could have deep conseguences in cooling a cylinder by a radial heat flow through superfluid helium, because $|\dot{Q}_{out}|$ small means that the heat removal would be very low, in contrast to predictions based on Fourier's theory. This does not mean that in usual practical situations the value of (10) is high or the value of R is low. The most relevant point is that considering rectification is a stimulus for a deeper understanding of the several nonlinear factors contributing to inwards and outwards heat flow, which are of natural interest in cryogenic.

Other situations worth of analysis would be **a)** to explore heat rectification for relatively small values of the quantum Reynolds numbers, in such a way that turbulence is not completely developed; **b)** to consider heat rectification in longitudinal channels with varying cross section; or **c)** to analyze heat rectification in longitudinal flows in troncoconical channels by considering the reversal heat transport regimes (laminar, turbulent, laminar–to–turbulent, laminar–to–ballistic).

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