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TWO SIMPLE WAYS TO FIND AN EFFICIENT SOLUTION FOR A MULTIPLE OBJECTIVE LINEAR PROGRAMMING PROBLEM

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ABSTRACT

A number of methods and techniques for determining "effective" solutions for multiple objective linear programming problems (MPP) have been developed. In this study, we will present two simple methods for determining an efficient solution for a MPP that reducing the given problem to a one-objective linear programming problem. One of these methods falls under the category of methods of weighted metrics, and the other is an approach similar to the ε – constraint method. The solutions determined by the two methods are not only effective and are found on the Pareto frontier, but are also "the best" in terms of distance to the optimal solutions for all objective function from the MPP. Obviously, besides the optimal solutions of linear programming problems in which we take each objective function, we can also consider the ideal point and Nadir point, in order to take into account all the notions that have been introduced to provide a solution to this problem.

KEYWORDS:

Multiple objective linear programming problem (MPP), efficient solution, Pareto front, methods of weighted metrics, ε – constraint method

1. Introduction

A *multiple objective linear programming problem* (MPP) has the following form:

$$MPP: \begin{cases} min\ z_1 = c^1 \cdot x \\ min\ z_2 = c^2 \cdot x \\ \dots \\ min\ z_r = c^r \cdot x \\ A \cdot x \leq b \\ x \geq 0 \end{cases}, \qquad (1)$$

where $c^1, c^2, \dots, c^r \in \boldsymbol{R}^n$, $A \in M_{m,n}$ ($\boldsymbol{R}$), with $m \leq n$ and rank $A = m$, are known and $x \in \boldsymbol{R}$ is the unknown vector that has to be determined. Of course, as it is mentioned in other materials, there is no restriction considering the form (1) for a MPP, the degree of generality being kept.

Usually, the optimum solutions for the individual *linear programming problems* (LLP) of the problem (1),

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$$\begin{cases} min\ z_i = c^i \cdot x \\ A \cdot x \leq b \qquad , i \in \overline{1,r}\ , \\ x \geq 0 \end{cases} \tag{2}$$

are different, and the idea is to find the solution that "harmonizes" all the objective functions of a MPP.

Because there is no optimal solution for all objective functions, the approach to a MPP must be done by introducing concepts that allow the comparison of the admissible solutions, and in this way we can also find "optimal solutions" in this case. The fundamental difference to the single objective optimization, where a total ordering of the solutions exists, is the fact that two admissible solutions can be incomparable.

The fundamental issue for an MPP is the determination of all admissible solutions that can not be compared by constructing methods and techniques to determine, if not all of the solutions mentioned above, at least one or a part of them.

There are different approaches that lead to the construction of methods and techniques to find such solutions. In this respect, we can mention the linear combination of weights, the scalarization method (method of weighted metrics), Geoffrion-Dyer-Feinberg method, Tchebycheff method, ε – constraint method, k^{th} – objective weighted constraint problem, lexicographic method, reference point methods recently multiobjective evolutionary algorithms. Aspects about these methods can be found in (Ehrgott, 2005; Narzisi, 2008; Antonio, Saul and Carlos, 2009; Dubois-Lacoste, Lopez-Ibanez and Stutzle, 2013 Scap, Hoic and Jokic, 2013).

In the following, we will recall some concepts with which we will continue to operate.

2. Basic concepts

Let f: $R^n \rightarrow R^r$, $f(x) = (f_1(x), f_2(x), \ldots, f_k(x), \ldots, f_r(x))$ a function which could be considered the actual function made up of all objective functions.

With regard to this function we will use the following concepts, which are also introduced in (Ehrgott, 2005; Narzisi, 2008; Antonio, Saul and Carlos, 2009; Dubois-Lacoste, Lopez-Ibanez and Stutzle, 2013; Scap, Hoic and Jokic, 2013):

Definition 1

A solution x_1 is said to *dominate* a solution x_2 (we are writing $x_1 < x_2$) if and only if $f_k(x_1) \leq f_k(x_2)$, $(\forall)$ $k = 1, 2, \ldots, r$ and $(\exists)$ $j \in \{1, 2, \ldots, r\}$ such that $f_j(x_1) < f_j(x_2)$.

Definition 2

A solution x_1 is said to *weakly dominate* a solution x_2 (we are writing $x_1 \leq x_2$) if and only if $f_k(x_1) \leq f_k(x_2)$, $(\forall)$ $k = 1, 2, \ldots, r$.

Definition 3

Solutions x_1 and x_2 are said to be *incomparable* ($x_1 \,||\, x_2$) if and only if neither $x_1 \npreceq x_2$ nor $x_2 \npreceq x_1$, and $x_1 \neq x_2$.

Definition 4

Let $P = \{x \in R^n \,/\, A \cdot x \leq b,\ x \geq 0\}$ denotes the set of all feasible solutions. A solution $x_1 \in P$ is a *Pareto global optimum* if and only if $(\nexists)$ $x_2 \in P$ such that $x_2 < x_1$. Such solutions are also called *efficient*.

Definition 5

A set X of solutions is a Pareto local optimum if $(\forall)$ $x \in X$, $(\forall)$ $x_1 \in N(x)$, $(\exists)$ $x_2 \in X$ verifying $x_2 \leq x_1$, where $N(x)$ is the set of all neighbors of solution x.

Definition 6

If P denotes the set of all feasible solutions, then a set P' is a Pareto global optimum set if and only if it contains all the Pareto global optimum solutions of P and only these solutions. The set of objective vectors of the Pareto global optimum is called the *Pareto front*.

The problem of determining the *Pareto front* for a MPP is a complicated and important one, in (Scap, Hoic and Jokic, 2013) being presented a method of analytical determination (from an equation), but for a MPP with only two objective functions.

In the next paragraph we will determine at least one effective solution for a MPP by using a scalarization method.

3. Method 1

The first method consists in building a linear programming problem with a single objective function that sums all the objective functions of problem 1, namely which is like

$$\begin{cases} \min z = \sum_{k=1}^{r} c^k \cdot x \\ A \cdot x \le b \\ x \ge 0 \end{cases} \quad . \qquad (3)$$

Problem (3) is equivalent to a problem of the following form

$$\begin{cases} \min z = \sum_{k=1}^{r} \left| c^k \cdot x - y_k^I \right| \\ A \cdot x \le b \\ x \ge 0 \end{cases} \quad , \qquad (4)$$

because

$$\sum_{k=1}^{r} \left| c^k \cdot x - y_k^I \right| = \sum_{k=1}^{r} \left(c^k \cdot x - y_k^I \right) =$$

$\sum_{k=1}^{r} c^k \cdot x - \sum_{k=1}^{r} y_k^I$, the minimum of this function is obtained when the minimum function $\sum_{k=1}^{r} c^k \cdot x$ is obtained, the quantity $\sum_{k=1}^{r} y_k^I$ being constant.

Let MPP proposed by Woody, but with coefficients divided by 10:

$$\begin{cases} \max z_1 = 5 \cdot x_1 + 7 \cdot x_2 + 8 \cdot x_3 \\ \min z_2 = 4 \cdot x_1 + 2 \cdot x_2 + x_3 \\ 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \le 30 \\ \qquad 3 \cdot x_2 + 4 \cdot x_3 \le 12 \\ x_1 + 2 \cdot x_2 + 3 \cdot x_3 \le 16 \\ x_1,\ x_2,\ x_3 \ge 0 \end{cases} \quad . \qquad (5)$$

With the proposed method, in order to determine an efficient solution of the problem (5), we will determine the optimal solutions of the problem

$$\begin{cases} \min z = -x_1 - 5 \cdot x_2 - 7 \cdot x_3 \\ 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \le 30 \\ \qquad 3 \cdot x_2 + 4 \cdot x_3 \le 12 \\ x_1 + 2 \cdot x_2 + 3 \cdot x_3 \le 16 \\ x_1,\ x_2,\ x_3 \ge 0 \end{cases} \quad . \qquad (6)$$

The standard form of the problem (6) is

$$\begin{cases} \min z = -x_1 - 5 \cdot x_2 - 7 \cdot x_3 \\ 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 + x_4 = 30 \\ \qquad 3 \cdot x_2 + 4 \cdot x_3 + x_5 = 12. \\ x_1 + 2 \cdot x_2 + 3 \cdot x_3 + x_6 = 16 \\ x_i \ge 0,\ (\forall) i \in \overline{1,6} \end{cases} \qquad (7)$$

The simplex table corresponding to it is:

Table no. 1

Simplex table for problem (7)

B	c_B	$\bar{x}^B$	-1	-5	-7	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	
a^4	0	30	2	3	4	1	0	0	7 1/2
a^5	0	12	0	3	4	0	1	0	3
a^6	0	16	1	2	3	0	0	1	5 1/3
z_j		0	0	0	0	0	0	0	–
$z_j - c_j$		–	1	5	7*	0	0	0	–
a^4	0	18	2	0	0	1	−1	0	9
a^3	−7	3	0	3/4	1	0	1/4	0	–
a^6	0	7	1	−1/4	0	0	−3/4	1	7
z_j		−21	0	−5 1/4	−7	0	−1 3/4	0	–
$z_j - c_j$		–	1	−1/4	0	0	−1 3/4	0	–
a^4	0	4	0	1/2	0	1	1/2	−2	
a^3	−7	3	0	3/4	1	0	1/4	0	
a^1	−1	7	1	−1/4	0	0	−3/4	1	
z_j		−28	−1	−5	−7	0	−1	−1	–
$z_j - c_j$		–	0	0	0	0	−1	−1	–

We can observe that for this choice of values corresponding to the input and output criteria from the base, the optimal solution of the problem (7) is $x_1 = 7$, $x_2 = 0$, $x_3 = 3$, $x_4 = 4$, $x_5 = 0$ and $x_6 = 0$ and consequently the efficient solution of the problem (5) is $x_1 = 7$, $x_2 = 0$ and $x_3 = 3$, a solution that is not dominated by any other efficient solution on the Pareto frontier.

Table no. 2

Simplex table for problem (7)

B	c_B	$\bar{x}^B$	-1	-5	-7	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	
a^4	0	30	2	3	4	1	0	0	10
a^5	0	12	0	3	4	0	1	0	4
a^6	0	16	1	2	3	0	0	1	8
z_j		0	0	0	0	0	0	0	–
$z_j - c_j$		–	1	5*	7	0	0	0	–
a^4	0	18	2	0	0	1	−1	0	9
a^2	−5	4	0	1	1 1/3	0	1/3	0	–
a^6	0	8	1	0	1/3	0	−2/3	1	8
z_j		−20	0	−5	−6 2/3	0	−1 2/3	0	–
$z_j - c_j$		–	1	0	1/3	0	−1 2/3	0	–
a^4	0	2	0	0	−2/3	1	1/3	−2	
a^2	−5	4	0	1	1 1/3	0	1/3	0	
a^1	−1	8	1	0	1/3	0	−2/3	1	
z_j		−28	−1	−5	−7	0	−1	−1	–
$z_j - c_j$		–	0	0	0	0	−1	−1	–

With this choice of values corresponding to the input and output criteria from the base, the optimal solution of the problem (7) is $x_1 = 8$, $x_2 = 4$, $x_3 = 0$, $x_4 = 2$, $x_5 = 0$ and $x_6 = 0$, and the efficient solution of the problem (5) is $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, a solution that is not dominated by any other efficient solution on the Pareto frontier.

In conclusion, the efficient solutions corresponding to the problem (5), determined using this method are $x_1 = 7$, $x_2 = 0$ and $x_3 = 3$ and $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, which are "the best efficient solutions", that are not dominated by any other efficient solution on the Pareto frontier.

4. Method 2

The second method is a method that combines a scalarization method and a ε – constraint method and consists in building a linear programming problem with a single objective function that minimizes the sum of the deviations of the proposed solution from optimal solutions to problems of linear programming in which each objective is taken in turn.

Let $\overline{x}^{B_i}$, $i \in \overline{1,r}$, the optimal solutions of problems (2) and $a_i = min\ z_i = c^i \cdot \overline{x}^{B_i}$. We will make the following linear programming problem that will lead us to the "best solutions" that are not dominated by any other efficient solution on the Pareto frontier.

$$\begin{cases} min\ z = \sum_{k=1}^{r} c^k \cdot x + \sum_{k=1}^{r} \varepsilon_i \\ z_i + \varepsilon_i \geq a_i \\ A \cdot x \leq b \\ x \geq 0,\ \varepsilon_i \geq 0,\ i \in \overline{1,r} \end{cases} . \tag{8}$$

With this method proposed to determine an effective solution to problem (5), we will determine the optimal solutions to the problem

$$\begin{cases} min\ z = -x_1 - 5 \cdot x_2 - 7 \cdot x_3 + \varepsilon_1 + \varepsilon_2 \\ 5 \cdot x_1 + 7 \cdot x_2 + 8 \cdot x_3 + \varepsilon_1 \leq 75 \\ 4 \cdot x_1 + 2 \cdot x_2 + x_3 + \varepsilon_2 \geq 0 \\ 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \leq 30. \\ 3 \cdot x_2 + 4 \cdot x_3 \leq 12 \\ x_1 + 2 \cdot x_2 + 3 \cdot x_3 \leq 16 \\ x_1,\ x_2,\ x_3,\ \varepsilon_1, \varepsilon_2 \geq 0 \end{cases} \tag{9}$$

The standard form of the problem (9) is

$$\begin{cases} min\ z = -x_1 - 5 \cdot x_2 - 7 \cdot x_3 + \varepsilon_1 + \varepsilon_2 \\ 5 \cdot x_1 + 7 \cdot x_2 + 8 \cdot x_3 + \varepsilon_1 + x_4 = 75 \\ -4 \cdot x_1 - 2 \cdot x_2 - x_3 - \varepsilon_2 + x_5 = 0 \\ 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 + x_6 = 30. \\ 3 \cdot x_2 + 4 \cdot x_3 + x_7 = 12 \\ x_1 + 2 \cdot x_2 + 3 \cdot x_3 + x_8 = 16 \\ x_i \geq 0,\ (\forall) i \in \overline{1,8},\ \varepsilon_1, \varepsilon_2 \geq 0 \end{cases} \tag{10}$$

The simplex table corresponding to it is:

Table no. 3

Simplex table for problem (10)

B	c_B	$\overline{x}^B$	−1	−5	−7	1	1	0	0	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	
a^6	0	75	5	7	8	1	0	1	0	0	0	0	9 3/8
a^7	0	0	−4	−2	−1	0	−1	0	1	0	0	0	–
a^8	0	30	2	3	4	0	0	0	0	1	0	0	7 1/2
a^9	0	12	0	3	4	0	0	0	0	0	1	0	3
a^{10}	0	16	1	2	3	0	0	0	0	0	0	1	5 1/3
z_j		0	0	0	0	0	0	0	0	0	0	0	–
$z_j - c_j$		–	1	5	7	−1	−1	0	0	0	0	0	–
a^6	0	51	5	1	0	1	0	1	0	0	−2	0	10 1/5
a^7	0	3	−4	−1 1/4	0	0	−1	0	1	0	1/4	0	–
a^8	0	18	2	0	0	0	0	0	0	1	−1	0	9
a^3	−7	3	0	3/4	1	0	0	0	0	0	1/4	0	–
a^{10}	0	7	1	− 1/4	0	0	0	0	0	0	−3/4	1	7
z_j		−21	0	−5 1/4	−7	0	0	0	0	0	−1 3/4	0	–
$z_j - c_j$		–	1	− 1/4	0	−1	−1	0	0	0	−1 3/4	0	–
a^6	0	16	0	2 1/4	0	1	0	1	0	0	1 3/4	−5	
a^7	0	31	0	−2 1/4	0	0	−1	0	1	0	−2 3/4	4	
a^8	0	4	0	1/2	0	0	0	0	0	1	1/2	−2	
a^3	−7	3	0	3/4	1	0	0	0	0	0	1/4	0	
a^1	−1	7	1	−1/4	0	0	0	0	0	0	− 3/4	1	
z_j		−28	−1	−5	−7	0	0	0	0	0	−1	−1	–
$z_j - c_j$		–	0	0	0	−1	−1	0	0	0	−1	−1	–

We can observe that, in this case, for this choice of the values corresponding to the input and output criteria from the base, the optimal solution of problem (10) is $x_1 = 7$, $x_2 = 0$, $x_3 = 3$, $x_6 = 16$, $x_7 = 31$, $x_8 = 4$ and $x_2 = x_4 = x_5 = x_9 = x_{10} = 0$, and the efficient solution of problem (5) is $x_1 = 7$, $x_2 = 0$ and $x_3 = 3$, a solution which is not dominated by any other effective solution on Pareto frontier.

Table no. 4

Simplex table for problem (10)

B	c_B	$\overline{x}^{B}$	−1	−5	−7	1	1	0	0	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	
a^6	0	75	5	7	8	1	0	1	0	0	0	0	10 5/7
a^7	0	0	−4	−2	−1	0	−1	0	1	0	0	0	–
a^8	0	30	2	3	4	0	0	0	0	1	0	0	10
a^9	0	12	0	3	4	0	0	0	0	0	1	0	4
a^{10}	0	16	1	2	3	0	0	0	0	0	0	1	8
z_j		0	0	0	0	0	0	0	0	0	0	0	–
$z_j - c_j$		–	1	5	7	−1	−1	0	0	0	0	0	–
a^6	0	47	5	0	−1 1/3	1	0	1	0	0	−2 1/3	0	9 2/5
a^7	0	8	−4	0	1 2/3	0	−1	0	1	0	2/3	0	–
a^8	0	18	2	0	0	0	0	0	0	1	−1	0	9
a^2	−5	4	0	1	1 1/3	0	0	0	0	0	1/3	0	–
a^{10}	0	8	1	0	1/3	0	0	0	0	0	−2/3	1	8
z_j		−20	0	−5	−6 2/3	0	0	0	0	0	−1 2/3	0	–
$z_j - c_j$		–	1	0	1/3	−1	−1	0	0	0	−1 2/3	0	–
a^6	0	7	0	0	−3	1	0	1	0	0	1	−5	
a^7	0	40	0	0	3	0	−1	0	1	0	−2	4	
a^8	0	2	0	0	−2/3	0	0	0	0	1	1/3	−2	
a^2	−5	4	0	1	1 1/3	0	0	0	0	0	1/3	0	
a^1	−1	8	1	0	1/3	0	0	0	0	0	−2/3	1	
z_j		−28	−1	−5	−7	0	0	0	0	0	−1	−1	–
$z_j - c_j$		–	0	0	0	−1	−1	0	0	0	−1	−1	–

With this choice of values corresponding to the input and output criteria from the base, the optimal solution of the problem (10) is $x_1 = 8$, $x_2 = 4$, $x_3 = 0$, $x_6 = 7$, $x_7 = 40$, $x_8 = 2$ and $x_3 = x_4 = x_5 = x_9 = x_{10} = 0$, and the efficient solution of problem (5) is $x_1 = 7$, $x_2 = 0$ and $x_3 = 3$, a solution which is not dominated by any other effective solution on Pareto frontier.

In conclusion, the efficient solutions corresponding to the problem (5), determined using this method are $x_1 = 7$, $x_2 = 0$ and $x_3 = 3$ and $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, which are the "best solutions" that are not dominated by any other efficient solution on the Pareto frontier.

5. Conclusions

The two proposed methods do not solve the problem of determining the Pareto frontier but only determine the "best solutions" on it, in practice this being the aspect that interests us. Ultimately, for choosing efficient Pareto border solutions for a MPP, it also comes back to aspects of different metrics that are being created to choose a compromise solution for all objective functions within the given MPP.

Obviously, the first method is simpler and leads faster to effective solutions of a MPP, but the second method virtually doubles this goal.

From solving problems (7) and (10) we note that we have to pay attention to the entry criterion in the base because if we choose to get out of the base a^k for which $z_k - c_k = max\ \{z_j - c_j \,/\, z_j - c_j > 0, j \in J_R\}$, J_R being the set of nonbasic indices, we risk losing out of the best LPP solutions.

It is an important aspect that we will be dealing with in a future study in which to build an IT application that takes this into account.

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