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ABOUT APPLICATIONS OF THE FIXED POINT THEORY

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ABSTRACT

The fixed point theory is essential to various theoretical and applied fields, such as variational and linear inequalities, the approximation theory, nonlinear analysis, integral and differential equations and inclusions, the dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimisation problems) and mathematical modelling. This paper presents a few benchmarks regarding the applications of the fixed point theory. This paper also debates if the results of the fixed point theory can be applied to the mathematical modelling of quality.

KEYWORDS:

Fixed point theory; game theory; applications; quality management

1. Introduction

The scientific basis of the fixed point theory was established in the 20th century. The fundamental result of this theory is the Picard-Banach-Caccioppoli contraction principle (from the '30s), which generated important lines of research and applications of the theory to functional equations, differential equations, integral equations, etc.

Classic theorems of this theory are the theorems of Tarki, Bourbaki, Banach, Perov, Luxemburg-Jung, Brower, Schauder, Tihonov, and Brouwder-Ghode-Kirk (Rus, Petruşel, Petruşel, 2008).

Banach's fixed point theory, also known as the contraction principle, is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of solutions to equations of the form $x = f(x)$, for a wide range of applications f, and it also provides a constructive method to determine these solutions. The theorem was created and demonstrated in the year 1922 by Stefan Banach (1892-1945), the founder of functional analysis.

The contraction principle is an abstract version of the successive approximation method; the method has been used empirically from antiquity, in order to solve numeric equations, and it has been successfully used, for instance, to solve Kepler's equation, $E = M + e \sin E$, in order to determine the position of the planets in orbit ($E_0=M$, $E_1=M+e\sin(E_0)$, ..., $E_n=M+e \sin(E_{n-1})$). Kepler's equations are used to calculate the position of objects from our Solar System, by using the eccentricity e of the orbit and the mean anomaly M. E represents the eccentric anomaly.

The successive approximations method was created by Joseph Liouville in

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the year 1837. It was later developed by Emile Picard in the year 1890.

Starting with the multivalued version of the Banach-Caccioppoli contraction principle, demonstrated by S. B. Nadler Jr. in 1969, the fixed point theory for multivalued operators in metric spaces has been used in many works published in the specialty literature. The development of this theory led to the development of various applications in numerous fields, such as: the optimisation theory, integral and differential equations and inclusions, the theory of fractals, econometrics, etc.

Among fixed point theorems with multivalued applications, one with remarkable applications is also the Avramescu-Markin-Nadler theorem.

Applications of the fixed point theorem that were identified and demonstrated were: Kasahara fixed point theorems, fixed point theorems for Darboux functions, etc.

Other important results with many applications to the fixed point theory are the fixed point theorems in fuzzy metric spaces, which are included in many works published in the specialty literature (i.e. Rao, Babu, Raju, 2009).

Fixed point theory is important not only in the existence of the theory of differential equations, integral equations, differential inclusions, integral inclusions, functional equations, partial differential equations, random differential equations (i.e. Rus, Petruşel, Petruşel, 2008; Longa, Nieto, Son, 2016), the approximation methods (i.e. Petruşel, Yao, 2009; Mishra, Pant, Panicker, 2016) but also in economics and management (i.e. Li, 2014), in computer science (i.e. Hasanzade Asl, Rezapour, Shahzad, 2012) and other domains (i.e. Isac,Yuan, Tan, Yu, 1998; Rus, Iancu, 2000; Song, Guo, Chen, 2016).

The first constructive method of calculating the fixed point of a continuous function was presented by H. Scarf in the year 1973.

The fixed point theory has applications in many problems, such as the existence of solutions, the existence of orbits in dynamical systems, in programming, etc. (Alfuraidan & Ansari, 2016).

The fixed points of certain important single-valued mappings also play an important role, as their results can be applied in engineering, physics, computer science, economics, and in telecommunication (Alfuraidan & Ansari, 2016).

A new research direction involves a new type of operator, namely the α-ψ-contractive type operator. Published works include, for instance the conditions of the *w*-distance. The issue with that is that it is difficult to find concrete examples for the *w*-distance, as it is a more abstract concept in itself. Asl, Rezapour, and Shahzad (2012) created the notion of α-ψ-contractive multivalued operators and demonstrated how to achieve fixed points results for this new type of operator. Guran and Bota (2015) studied in their paper the existence, uniqueness and generalised Ulam-Hyers stability of a fixed point of α-ψ-contractive type operator on a KST-space. A new problem is establishing conditions in which the fixed points of the α-ψ-contractive type operators would exist and would be unique, which is the problem of these types of contractions in vector spaces.

2. Fixed Point Theory Applied to the Game Theory. Particular Case – Games for the Field of Quality

The problems related to the analysis of the quality of a tangible or intangible product may be approached, in some cases, as problems from the game theory.

Creating a tangible or intangible product depends on two groups of factors: one group that increases the values of the product's quality indicators, and the other group that decreases the values of the product's quality indicators. Therefore, the first player is determined by the factors that increase the values of the quality indicators, and the second player by the other group of factors. The first player "wishes" to create a high quality product, whereas the second player "wishes" to create a poor quality product. The result of the competition between them is the actual quality of the product.

According to dedicated scientific notation, we denote by

$A_1 = \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m\}$ the set of factors that lead to an increase in the values of the quality indicators, and by

$A_2 = \{a_1, \ldots, a_j, \ldots, a_n\}$ the set of factors that lead to a decrease. At one moment of time from the life cycle of a tangible or intangible product, each player has a certain influence on the values of the quality indicators.

Each player chooses an action α_i from A_1 and a_j from A_2. The actions refer to the effect of factor i on the values of the quality indicators.

The utility of choosing action α_i by the first player can be described mathematically by a real function $f_1(\alpha_i, a_j)$ and its values can be interpreted as a win for the first player, in this situation. Function $f_2(\alpha_i, a_j)$ represents the second player's loss, in this situation.

According to the specialty literature, the fact that the sum of the game is null can be written as (Owen, 1974):

$$f_1(\alpha_i, a_j) + f_2(\alpha_i, a_j) = 0.$$

The question is how the first player can choose the action α_i in order to achieve a maximum gain $f_1(\alpha_i, a_j)$, knowing that the other player has the same objective (the term *utility* was introduced by von Neumann, and it significantly expanded the concept of "game", suggesting that a "result" of a game is not only something financial, but also a diverse multitude of events for which each player shows interest, quantified by their utility). The Nash Equilibrium in a pure strategy is represented by a strategic profile in which the strategy of each player is the best response to the strategy chosen by the other player. Thus, the conflict situations regarding creating tangible or intangible products of a high quality level and their management can be modelled by using the interconnections between these theories (Figure no.1):

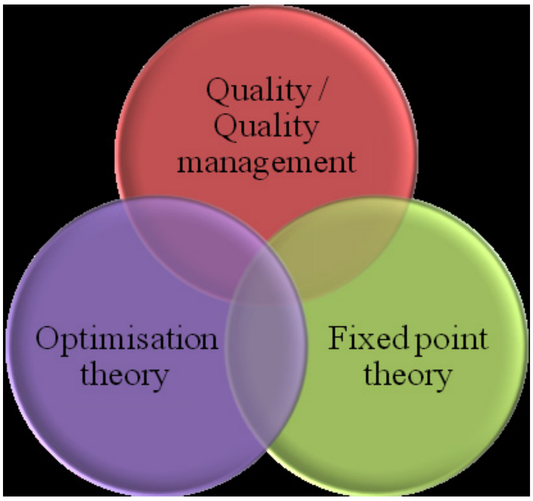

Figure no. 1 Illustrating the interconnection between the theories

Several authors (i.e. Yang and Yu, 2002; Yu and Yang, 2004; Lin, 2005) have demonstrated that the fixed point theory can be applied to optimisation problems, game theory problems, and also in problems related to the Nash equilibrium.

Example: The quality of the promotion service in universities U_1 and U_2, in order to increase the number of candidates that wish to enrol in their courses. Based on the data obtained by analysing the number of students that applied in the previous years, the marketing services of the two universities have reached the conclusions shown in the following table:

Table no. 1

Results

Strategy of U_1, generated by its set $A_1 = \{\alpha_1, \alpha_2\}$ of action possibilities	Strategy of U_2, generated by its set $A_2 = \{a_1, a_2\}$ of action possibilities	Result (change in the percentage of candidates that wish to enrol to the university)
α_1 (uses flyers)	**a_1 (uses flyers)**	**$f_1(\alpha_i, a_j) = 10$ (U_1 wins), $f_2(\alpha_i, a_j) = -10$ (U_2 loses)**
α_1 (uses flyers)	**a_1 (uses advertising media)**	**$f_1(\alpha_i, a_j) = 6$ (U_1 wins), $f_2(\alpha_i, a_j) = -6$ (U_2 loses)**
α_2 (uses advertising media)	**a_2 (uses flyers)**	**$f_1(\alpha_i, a_j) = -12$ (U_1 loses), $f_2(\alpha_i, a_j) = 12$ (U_2 wins)**
α_2 (uses advertising media)	**a_2 (uses advertising media)**	**$f_1(\alpha_i, a_j) = 2$ (U_1 loses), $f_2(\alpha_i, a_j) = -2$ (U_2 wins)**

The results from table no. 1 can be summarised as seen in table no. 2:

Table no. 2

Summary of results

	a_1	a_2	
1	10	6	$min\{10, 6\} - 6$
2	-12	2	$min\{-12, 2\} = -12$
	$max\{10, 12\} - 10$	$max\{6, 2\} = 6$	$min\{10, 6\} - 6 - max\{6, -12\}$

The game has an equilibrium point, (α_i, a_j), which coincides with the result of the game. University U_1 will want to use action α_i, meaning to use the flyers, whereas university U_2 will try to minimise the loss by choosing action a_j.

Games can be cooperative and non-cooperative. In 1986, Kohlberg and Mertens proved that for every finite non-cooperative game, the set of Nash equilibrium points consists of finite components and at least one of them is essential. In 2004, Yu and Yang suggested that essential components can be applied to nonlinear problems.

In 2012, Vuong, Strodiot, and Nguyen introduce some new iterative methods for finding a common element of the set of points satisfying a Ky Fan inequality, and the set of fixed points of a contraction mapping in a Hilbert space.

3. Conclusions

The fixed point theory has had many applications in the last decades. Its applications are very useful and interesting to the optimisation theory, to the game theory, to conflict situations, but also to the mathematical modelling of quality and its management.

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