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NOTE ON THE SCHRÖDINGER EQUATION

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Abstract
A second-order formalism leading to an equation describing the same dynamics as the Schrödinger one is developed under some compatible initial conditions.

It is well-known that the Euler-Lagrange [1] and Hamilton [2] equations are involved in many aspects of theoretical physics. On the one hand, the Schrödinger equation [3]-[4] can be derived from the first-order Lagrangian

$$\Lambda_0 = \frac{i\hbar}{2}(\psi^* \dot{\psi} - \dot{\psi}^* \psi) - \frac{\hbar^2}{2m}(\partial_i \psi^*)(\partial_i \psi) - V\psi^* \psi. \tag{1}$$

On the other hand, the Hamiltonian formulation of the Schrödinger equation was involved in many applications of quantum mechanics [5]-[9].

In this paper we develop a second-order formalism leading to an equation that describes the same dynamics as the Schrödinger one under some compatible initial conditions. In the sequel, we restrict ourselves to the one-particle Schrödinger equation with a time independent potential $V(\mathbf{x})$.

From the canonical approach of (1), one infers the second-class constraints

$$\chi \equiv \pi - \frac{i\hbar}{2}\psi^* \approx 0,\ \chi^* \equiv \pi^* + \frac{i\hbar}{2}\psi \approx 0, \tag{2}$$

and the canonical Hamiltonian

$$H_0(t) = \int d^3x \left(\frac{\hbar^2}{2m}(\partial_i \psi^*)(\partial_i \psi) + V\psi^* \psi \right). \tag{3}$$

The notations π and π^* signify the canonical momenta conjugated with ψ, respectively ψ^*

$$[\psi(\mathbf{x},t),\pi(\mathbf{y},t)]=\delta^3(\mathbf{x}-\mathbf{y})=[\psi^*(\mathbf{x},t),\pi^*(\mathbf{y},t)], \tag{4}$$

where the symbol $[,]$ denotes the Poisson bracket. Thus, the Hamiltonian equations of motion can be written as

$$\dot{F}(\mathbf{x},t)=\left[F(\mathbf{x},t),H_0(t)\right]^{\bullet}, \tag{5}$$

where the Dirac bracket [10]-[12] takes the form

$$\begin{aligned}\left[F_1(\mathbf{x},t),F_2(\mathbf{y},t)\right]^{\bullet} &= \left[F_1(\mathbf{x},t),F_2(\mathbf{y},t)\right]-\\ &-\frac{i}{\hbar}\int d^3z[F_1(\mathbf{x},t),\chi(\mathbf{z},t)][\chi^*(\mathbf{z},t),F_2(\mathbf{y},t)]+\\ &+\frac{i}{\hbar}\int d^3z[F_1(\mathbf{x},t),\chi^*(\mathbf{z},t)][\chi(\mathbf{z},t),F_2(\mathbf{y},t)].\end{aligned} \tag{6}$$

After eliminating the second-class constraints (the independent co-ordinates of the reduced phase-space are ψ and ψ^*), with the help of (5) we find that the dynamics is governed by the equations of motion

$$\dot{\psi}=\frac{i\hbar}{2m}\partial_i\partial_i\psi-\frac{i}{\hbar}V\psi,\ \dot{\psi}^*=-\frac{i\hbar}{2m}\partial_i\partial_i\psi^*+\frac{i}{\hbar}V\psi^*, \tag{7}$$

which are nothing but the Schrödinger equations for ψ and ψ^*.

Now, we start with the Hamiltonian

$$\overline{H}_0(t)=\int d^3x\left(\pi^*+\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi-V\psi\right)\right)\left(\pi-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi^*-V\psi^*\right)\right) \tag{8}$$

from which we derive the Hamilton equations[1]

$$\dot{\psi}=\pi^*+\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi-V\psi\right), \tag{9}$$

$$\dot{\psi}^*=\pi-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi^*-V\psi^*\right), \tag{10}$$

$$\dot{\pi}=-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\left(\pi-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi^*-V\psi^*\right)\right), \tag{11}$$

$$\dot{\pi}^*=\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\left(\pi^*+\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi-V\psi\right)\right). \tag{12}$$

Regarding the equations (9-12) we choose the initial conditions[2]

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[1] It is easy to see that the Hamiltonian (8) describes a non-degenerate system.

[2] It is obvious that the initial conditions (13-14) imply the relations $\psi^*(\mathbf{x},t_0)=\psi_0^*(\mathbf{x})$, $\pi^*(\mathbf{x},t_0)=-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi_0(\mathbf{x})$.

$$\psi(\mathbf{x},t_0)=\psi_0(\mathbf{x}), \tag{13}$$

$$\pi(\mathbf{x},t_0)=-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi_0^*(\mathbf{x}). \tag{14}$$

Substituting (9) in (12) and (10) in (11) we derive the equations

$$\frac{\partial}{\partial t}\left(\pi^*-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi-V\psi\right)\right)=0, \tag{15}$$

$$\frac{\partial}{\partial t}\left(\pi+\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i\psi^*-V\psi^*\right)\right)=0, \tag{16}$$

which lead to

$$\pi^*(\mathbf{x},t)-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi(\mathbf{x},t)=k(\mathbf{x}), \tag{17}$$

$$\pi(\mathbf{x},t)+\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi^*(\mathbf{x},t)=k^*(\mathbf{x}), \tag{18}$$

where $k(\mathbf{x})$ and $k^*(\mathbf{x})$ are some functions determined by the initial conditions. Writing down (17-18) for $t=t_0$ and using the initial conditions, we deduce the relations

$$k(\mathbf{x})=0=k^*(\mathbf{x}), \tag{19}$$

such that (17-18) lead to

$$\pi^*=\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi,\ \pi=-\frac{i}{2\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi^*. \tag{20}$$

Inserting (20) in (9-10) we arrive at (7). In consequence, we have proved the next result: c_1) $(\psi(\mathbf{x},t),\psi^*(\mathbf{x},t),\pi(\mathbf{x},t),\pi^*(\mathbf{x},t))$ are solutions of equations (9-12) subject to the initial conditions (13-14) if and only if $(\psi(\mathbf{x},t),\psi^*(\mathbf{x},t))$ are solutions of equations (7) subject to the initial conditions (13).

It is easy to show that the Hamiltonian (8) comes from the non-degenerate second-order Lagrangian

$$\overline{\Lambda}_0=\dot{\psi}^*\dot{\psi}-\frac{i}{2\hbar}\dot{\psi}^*\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi+\frac{i}{2\hbar}\dot{\psi}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi^*, \tag{21}$$

which is different from that used in [13]. At the Lagrangian level the initial conditions (13-14) take the form

$$\psi(\mathbf{x},t_0)=\psi_0(\mathbf{x}), \tag{22}$$

$$\dot{\psi}(\mathbf{x},t_0)=\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\partial_i\partial_i-V\right)\psi_0(\mathbf{x}). \tag{23}$$

Due to the fact that the Lagrangian (21) is non-degenerate the following standard result holds: c_1) $(\psi(\mathbf{x},t),\psi^*(\mathbf{x},t))$ are solutions to the Euler-Lagrange equations $\delta\overline{\Lambda}_0/\delta\psi = 0$, $\delta\overline{\Lambda}_0/\delta\psi^* = 0$ subject to the initial conditions (22-23) if and only if $(\psi(\mathbf{x},t),\psi^*(\mathbf{x},t),\pi(\mathbf{x},t),\pi^*(\mathbf{x},t))$ are solutions of equations (9-12) in the presence of the initial conditions (13-14).

Thus, results c_1) and c_2) lead to th following conclusion: *the solutions to the Euler-Lagrange equations* $\delta\overline{\Lambda}_0/\delta\psi = 0$, $\delta\overline{\Lambda}_0/\delta\psi^* = 0$ *subject to the initial conditions (22-23) coincide with the solutions to the equations (7) corresponding to the initial conditions (13).*

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