DOI: 10.1515/awutm-2017-0017

DE GRUYTER OPEN

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LV, 2, (2017), 91– 117

Schatten Class Operators in $\mathcal{L}(L_a^2(\mathbb{C}_+))$

Namita Das and Jitendra Kumar Behera

Abstract. In this paper, we consider Toeplitz operators defined on the Bergman space $L_a^2(\mathbb{C}_+)$ of the right half plane and obtain Schatten class characterization of these operators. We have shown that if the Toeplitz operators $\mathcal{T}_\phi$ on $L_a^2(\mathbb{C}_+)$ belongs to the Schatten class $S_p, 1 \leq p < \infty$, then $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$, where $\widetilde{\phi}(w) = \langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$ and $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{(s+w)^2}$. Here $d\nu(w) = |B(\overline{w}, w)| d\mu(w)$, where $d\mu(w)$ is the area measure on $\mathbb{C}_+$ and $B(\overline{w}, w) = (b_{\overline{w}}(\overline{w}))^2$. Furthermore, we show that if $\phi \in L^p(\mathbb{C}_+, d\nu)$, then $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ and $\mathcal{T}_\phi \in S_p$. We also use these results to obtain Schatten class characterizations of little Hankel operators and bounded operators defined on the Bergman space $L_a^2(\mathbb{C}_+)$.

AMS Subject Classification (2010). 47B10; 47B35; 30H20
Keywords. Right half plane; Bergman space; Schatten class; Toeplitz operators; little Hankel operators

1 Introduction

Let H be a separable Hilbert space. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself and $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$. For any nonnegative integer n, the nth singular value of $T \in \mathcal{LC}(H)$ is given by

$$s_n(T) = \inf\{||T - K|| : K \in \mathcal{L}(\mathrm{H}),\ \text{rank } K \leq n\}.$$

Here $||.||$ is the operator norm. Clearly, $s_0(T) = ||T||$ and

$$s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0. \tag{1.1}$$

For $0 < p < \infty$, the Schatten p-class ([16], [14]) of H, denoted by $S_p(H)$ or simply S_p, is defined as the space of all compact operators T on H with its singular value sequence $\{s_n\}_{n=1}^{\infty}$ belonging to l^p(the p-summable sequence space). If $1 \leq p < \infty$, the vector space S_p is a Banach space when equipped with the norm

$$||T||_p = \left(\sum_{n=1}^{\infty} |s_n|^p\right)^{\frac{1}{p}}.$$

The space S_1 is called the trace class and S_2 is the Hilbert-Schmidt class. For basic properties of Schatten class operators one can refer ([9], [17], [18], [4]) . The linear functional trace is defined on S_1 by

$$tr(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \xi_n\rangle, \quad T \in S_1,$$

where $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal basis for H. Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : x > 0\}$ be the right half plane. Let $d\mu(s) = dxdy$ denote the two dimensional area measure on $\mathbb{C}_+$. Let $L^2(\mathbb{C}_+, d\mu)$ be the space of complex-valued, absolutely square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. The Bergman space of the right half plane denoted as $L^2_a(\mathbb{C}_+)$ is the closed subspace of $L^2(\mathbb{C}_+, d\mu)$ consisting of those functions in $L^2(\mathbb{C}_+, d\mu)$ that are analytic. The functions $H(s, w) = \frac{1}{(s+\overline{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$ are the reproducing kernels [3] for $L^2_a(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. Define for $f \in L^\infty(\mathbb{C}_+), ||f||_\infty = \operatorname{ess}\sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm. For $\phi \in L^\infty(\mathbb{C}_+)$, we define the multiplication operator $\mathcal{M}_\phi$ from $L^2(\mathbb{C}_+, d\mu)$ into $L^2(\mathbb{C}_+, d\mu)$ by $(\mathcal{M}_\phi \mathrm{f})(s) = \phi(s)f(s)$; the Toeplitz operator $\mathcal{T}_\phi$ from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ by $\mathcal{T}_\phi$f$= P_+(\phi f)$, where P_+ denote the orthogonal projection from $L^2(\mathbb{C}_+, d\mu)$ onto $L^2_a(\mathbb{C}_+)$. The Toeplitz operator $\mathcal{T}_\phi$ is bounded and $||\mathcal{T}_\phi|| \leq ||\phi||_\infty$. For more details see [8] and [11]. The big Hankel operator $\mathcal{H}_\phi$ from $L^2_a(\mathbb{C}_+)$ into $(L^2_a(\mathbb{C}_+))^\perp$ is defined by $\mathcal{H}_\phi$f$= (I - P_+)(\phi f), f \in L^2_a(\mathbb{C}_+)$. For $\phi \in L^\infty(\mathbb{C}_+)$, the little Hankel operator $\hbar_\phi$ is a mapping from $L^2_a(\mathbb{C}_+)$ into $\overline{L^2_a(\mathbb{C}_+)}$ defined by $\hbar_\phi f = \overline{P}_+(\phi f)$, where $\overline{P}_+$ is the projection operator from $L^2(\mathbb{C}_+, d\mu)$ onto $\overline{L^2_a(\mathbb{C}_+)} = \{\overline{f} : f \in L^2_a(\mathbb{C}_+)\}$. There are also many equivalent ways for defining little Hankel operators on $L^2_a(\mathbb{C}_+)$. Let $\mathcal{S}_\phi$ be the mapping from $L^2_a(\mathbb{C}_+)$ into $L^2_a(\mathbb{C}_+)$ defined by $\mathcal{S}_\phi$f$= P_+(J(\phi f))$ where J is the mapping from

$L^2(\mathbb{C}_+, d\mu)$ into $L^2(\mathbb{C}_+, d\mu)$ such that $Jf(s) = f(\overline{s})$. Notice that J is unitary and $J\mathcal{S}_\phi$f$= J(P_+(J(\phi f))) = JP_+J(\phi f) = \overline{P}_+(\phi f) = \hbar_\phi f$ for all $f \in L_a^2(\mathbb{C}_+)$. Let $\mathbf{\Gamma}_\phi$ be the mapping from $L_a^2(\mathbb{C}_+)$ into $L_a^2(\mathbb{C}_+)$ defined by $\mathbf{\Gamma}_\phi$f$= P_+\mathcal{M}_\phi$Jf. Thus $\mathbf{\Gamma}_\phi f = P_+\mathcal{M}_\phi$Jf$= P_+(\phi(s)f(\overline{s})) = P_+(J(\phi(\overline{s})f(s))) = \mathcal{S}_{J\phi}$f for all $f \in L_a^2(\mathbb{C}_+)$. Hence $\mathbf{\Gamma}_\phi$f$= \mathcal{S}_{J\phi}$f. Thus we obtain $\hbar_\phi = J\mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi = \mathcal{S}_{J\phi}$. Since J is unitary, the three operators $\hbar_\phi, \mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi$ are referred to as little Hankel operators on $L_a^2(\mathbb{C}_+)$ and a given result on little Hankel operators can be stated using the operators $\hbar_\phi, \mathcal{S}_\phi$ and $\mathbf{\Gamma}_\phi$. The operator $\hbar_\phi$ is unbounded in general. However, $\hbar_\phi$ is bounded if $\phi \in L^\infty(\mathbb{C}_+)$ and we clearly have $||\hbar_\phi|| \leq ||\phi||_\infty$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D}, dA)$ be the space of complex valued, square-integrable, measurable functions on $\mathbb{D}$ with respect to the normalized area measure $dA(z) = \frac{1}{\pi}dxdy$. Let $L_a^2(\mathbb{D})$ be the space consisting of those functions of $L^2(\mathbb{D}, dA)$ that are analytic. The space $L_a^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$ and is called the Bergman space of the open unit disk $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ with the essential supremum norm. For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ from $L^2(\mathbb{D}, dA)$ into $L^2(\mathbb{D}, dA)$ is defined by $M_\phi f = \phi f$, the Toeplitz operator T_ϕ from $L_a^2(\mathbb{D})$ into itself is defined by $T_\phi(f) = P(\phi f)$ for $f \in L_a^2(\mathbb{D})$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. The sequence of functions $\{e_n(z)\}_{n=0}^\infty = \{\sqrt{n+1}z^n\}_{n=0}^\infty$ form an orthonormal basis for $L_a^2(\mathbb{D})$. Since the point evaluation at $z \in \mathbb{D}$, is a bounded functional, there is a function K_z in $L_a^2(\mathbb{D})$ such that $f(z) = \langle f, K_z\rangle$ for all f in $L_a^2(\mathbb{D})$. Let $K(z, w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, w) = \overline{K_z(w)}$. The function $K(z, w) = \frac{1}{(1-z\overline{w})^2}, z, w \in \mathbb{D}$ and is called the Bergman reproducing kernel [19]. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\overline{a}z)^2}$. The function $k_a, a \in \mathbb{D}$ is called the normalized reproducing kernel for $L_a^2(\mathbb{D})$.

It is not easy to verify that a linear operator is bounded, and it is even more difficult to determine its norm. No conditions on the matrix entries a_{ij} of an operator A have been found which are necessary and sufficient for A to be bounded, nor has $||A||$ been determined in the general case. For the more general problem we also need analogues of the notion of operator norm. For more details see ([19], [2]). The family of norms that has received much attention during the last decade is the Schatten norm. It is well known that [19] there are no compact Toeplitz operators on the Hardy space other than the zero operator. In the Bergman space setting, however, there are lots of nontrivial compact Toeplitz operators belonging to different Schatten classes. In view of this it is of interest to know the Schatten class characterizations of Toeplitz and Hankel operators defined on $L_a^2(\mathbb{C}_+)$. Such results play an important role in approximation theory [5]. Most of the results obtained so far on the Schatten class characterizations of Toeplitz and Hankel operators on the Bergman space of the disk (also on the Bergman space of the unit ball, weighted Bergman spaces of the disk and on bounded symmetric domains) are through the Berezin symbols of the corresponding operators. On the Bergman space of the disk [19], the Schatten class characterizations of big Hankel operators are given with the help of mean oscillation functions [19]. In the literature, the Schatten class characterizations of little Hankel operators are obtained in terms of a function

$$(Vf)(z) = 3\langle \bar{k}_z, h_{\bar{f}} k_z \rangle,$$

where $h_{\bar{f}}$ is the little Hankel operators on the respective Bergman space and k_z is the corresponding normalized reproducing kernel.

In this paper, we have shown that the functions $b_{\bar{w}}(s), B(s,w)$ and $B_{\bar{w}}(s)$ as defined in §2 will play vital role in obtaining the Schatten class characterizations for Toeplitz, big Hankel and little Hankel operators on $L^2_a(\mathbb{C}_+)$.

The layout of this paper is as follows. In §2, we introduce a class of unitary operators defined on $L^2_a(\mathbb{C}_+)$ induced by the automorphisms $t_a(s)$ of $\mathbb{C}_+$. These class of unitary operators play a major role in obtaining the Schatten class characterization of Toeplitz operators defined on $L^2_a(\mathbb{C}_+)$. In §3, we introduce the functions $B(s,w), B_{\overline{w}}(s)$ and $b_{\overline{w}}(s), s, w \in \mathbb{C}_+$ and establish relations between them. These functions will play a crucial role in obtaining the Schatten class characterizations of Toeplitz operators. In §4, we relate Toeplitz operators defined on $L^2_a(\mathbb{D})$ and $L^2_a(\mathbb{C}_+)$. The symbol correspondence is also given. We show that span $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L^2_a(\mathbb{C}_+)$ and prove that $\widetilde{|\phi|^2}(w) - |\widetilde{\phi}(w)|^2 \leq ||\mathcal{H}_\phi|| + ||\mathcal{H}_{\overline{\phi}}||$, where $\widetilde{\phi}(w) = \langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$. In §5, we prove that if A_1 is an operator in the trace class of $L^2_a(\mathbb{C}_+)$ then $tr(A_1) = \int_{\mathbb{C}_+} \widetilde{A}_1(w) d\nu(w)$ where $\widetilde{A}_1(w) = \langle A_1 b_{\overline{w}}, b_{\overline{w}} \rangle$. We also obtain the Schatten class characterization of positive Toeplitz operators. In §6, we present conditions to describe Schatten class Toeplitz operators on the Bergman space $L^2_a(\mathbb{C}_+)$ of the right half plane. In §7, we find conditions on $C \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ such that $C \in S_p$, the Schatten p-class, $1 \leq p < \infty$ by comparing with positive Toeplitz operators defined the Bergman space $L^2_a(\mathbb{C}_+)$ and applications of the result are also obtained. In §8, we show that the Schatten class properties of the little Hankel operator $\hbar_{\overline{f}} = J\mathcal{S}_{\overline{f}}, f \in L^2(\mathbb{C}_+, d\mu)$ depends only on the anti-analytic part of the symbol and establish that for $2 \leq p < \infty$, the little Hankel operator $\mathcal{S}_{\overline{\phi}} \in S_p$ if and only if $\mathcal{V}\phi \in L^p(\mathbb{C}_+, d\nu)$ where $d\nu(w) = |B(\overline{w}, w)| d\mu(w)$.

2 A class of unitary operators on $L_a^2(\mathbb{C}_+)$

In this section, we introduce a class of unitary operators defined on $L_a^2(\mathbb{C}_+)$. These class of unitary operators play a major role in obtaining the Schatten class characterization of Toeplitz operators defined on $L_a^2(\mathbb{C}_+)$.

Lemma 2.1. *If $a \in \mathbb{D}$ and $a = c + id$, $c, d \in \mathbb{R}$, then the following hold:*

(i) $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$.

(ii) $(t_a \circ t_a)(s) = s$.

(iii) $t'_a(s) = -l_a(s)$, where $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$.

Proof. This can be verified by direct calculations. □

For $a \in \mathbb{D}$, define $V_a : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{C}_+)$ by $(V_a g)(s) = (g \circ t_a)(s) l_a(s)$. In Proposition 2.2, we show that V_a is a self-adjoint, unitary operator which is also an involution.

Proposition 2.2. *For $a \in \mathbb{D}$,*

(i) $V_a l_a = 1$.

(ii) $V_a^{-1} = V_a$ and V_a is an involution, i.e. $V_a^2 = I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$, where $I_{\mathcal{L}(L_a^2(\mathbb{C}_+))}$ is an identity operator from $L_a^2(\mathbb{C}_+)$ into $L_a^2(\mathbb{C}_+)$.

(iii) V_a is self-adjoint.

(iv) V_a is unitary, $||V_a|| = 1$.

(v) $V_a P_+ = P_+ V_a$.

Proof. The proposition follows from the definition of V_a. □

.

Proposition 2.3. *Let $a \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{C}_+)$. Then $V_a \mathfrak{T}_\phi V_a = \mathfrak{T}_{\phi \circ t_a}$.*

Proof. Notice that since $(l_a \circ t_a)(s) l_a(s) = s$, we have for $f \in L_a^2(\mathbb{C}_+)$,

$$\begin{aligned} V_a \mathfrak{T}_\phi V_a f &= V_a \mathfrak{T}_\phi [(f \circ t_a) l_a] \\ &= V_a P_+ [\phi (f \circ t_a) l_a] \\ &= P_+ V_a [\phi (f \circ t_a) l_a] \\ &= P_+ [(\phi \circ t_a) f (l_a \circ t_a) l_a] \\ &= P_+ [(\phi \circ t_a) f] \\ &= \mathfrak{T}_{\phi \circ t_a} f. \end{aligned}$$

□

3 The function $B(s,w)$

In this section, we introduce the functions $B(s,w), B_{\overline{w}}(s)$ and $b_{\overline{w}}(s), s, w \in \mathbb{C}_+$ and establish relations between them. These functions will play a crucial role in obtaining the Schatten class characterizations of Toeplitz operators.

Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms)\frac{1}{(1+s)^2}$, where $Ms = \frac{1-s}{1+s}$. The map W is one-one and onto. Hence W^{-1} exists and $W^{-1} : L^2_a(\mathbb{C}_+) \to L^2_a(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi}G(Mz)\frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$. Suppose $a \in \mathbb{D}$ and $w = \frac{1-\overline{a}}{1+\overline{a}} = M\overline{a} \in \mathbb{C}_+$. Define $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}}\frac{1+w}{1+\overline{w}}\frac{2Rew}{(s+w)^2}$. Let $B(s,w) = B_{\overline{w}}(s) = \frac{1}{\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}\frac{1}{(1+s)^2}$.

Lemma 3.1. *Let $s, w \in \mathbb{C}_+$. The following hold:*

(i) $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$.

(ii) $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$.

Proof. Let $w \in \mathbb{C}_+$ and $w = M\overline{a} = \frac{1-\overline{a}}{1+\overline{a}}$. Notice that

$$\begin{aligned} b_{\overline{w}}(s) &= \frac{1}{\sqrt{\pi}}\frac{1+w}{1+\overline{w}}\frac{2Rew}{[s+w]^2} \\ &= \frac{2}{\sqrt{\pi}}\frac{1-|a|^2}{[1-\overline{a}(Ms)]^2}\frac{1}{(1+s)^2}, \end{aligned}$$

and hence

$$b_{\overline{w}}(\overline{w}) = \frac{1}{2\sqrt{\pi}}\frac{(1+a)^2}{(1-|a|^2)}.$$

Thus

$$b_{\overline{w}}(s)b_{\overline{w}}(\overline{w}) = \frac{(-1)}{2\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}M' = B(s,w).$$

Thus $b_{\overline{w}}(s) = \frac{B(s,w)}{b_{\overline{w}}(\overline{w})}$ and $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$. This proves (i). To prove (ii), notice that

$$\begin{aligned} ||B_{\overline{w}}||^2 &= \int_{\mathbb{C}_+} |B_{\overline{w}}(s)|^2 d\mu(s) \\ &= \int_{\mathbb{C}_+} |B(s,w)|^2 d\mu(s) \\ &= |b_{\overline{w}}(\overline{w})|^2, \end{aligned}$$

since $||b_{\overline{w}}||_2 = 1$. Thus $||B_{\overline{w}}|| = |b_{\overline{w}}(\overline{w})|$ and hence $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$. $\square$

4 Toeplitz and Hankel operators in $L_a^2(\mathbb{C}_+)$

In this section, we relate Toeplitz operators defined on $L_a^2(\mathbb{D})$ and $L_a^2(\mathbb{C}_+)$. The symbol correspondence is also given. We show that span $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L_a^2(\mathbb{C}_+)$ and prove that $\widetilde{|\phi|^2}(w) - |\widetilde{\phi}(w)|^2 \leq ||\mathcal{H}_{\phi}|| + ||\mathcal{H}_{\overline{\phi}}||$, where $\widetilde{\phi}(w) = \langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$.

Lemma 4.1. *Let $G(s) \in L^\infty(\mathbb{C}_+)$. Then the Toeplitz operator $\mathcal{T}_G$ defined on $L_a^2(\mathbb{C}_+)$ with symbol G is unitarily equivalent to the Toeplitz operator T_ϕ defined on $L_a^2(\mathbb{D})$ with symbol $\phi(z) = G\left(\frac{1-z}{1+z}\right)$, where $Mz = \frac{1-z}{1+z}$.*

Proof. The operator W maps $\sqrt{n+1}z^n$ to the function $\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n \frac{1}{(1+s)^2}$ which belongs to $L_a^2(\mathbb{C}_+)$. The Toeplitz operator $\mathcal{T}_G$ maps this vector to $P_+\left(G(s)\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n \frac{1}{(1+s)^2}\right)$ which is equal to

$$WPW^{-1}\left(G(s)\frac{2}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{1-s}{1+s}\right)^n \frac{1}{(1+s)^2}\right) = WP\left(G\left(\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\right)$$

$$= WT_\phi(z^n\sqrt{n+1}),$$

where $\phi(z) = G\left(\frac{1-z}{1+z}\right)$. Therefore $\mathcal{T}_G$ is unitarily equivalent to T_ϕ. $\square$

Lemma 4.2. *The space span $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L_a^2(\mathbb{C}_+)$.*

Proof. Suppose $g \in L_a^2(\mathbb{D})$ and g is orthogonal to $K_a, a \in \mathbb{D}$. Then $g(a) = \langle g, K_a \rangle = 0$ for all $a \in \mathbb{D}$, i.e. $g = 0$. Hence span $\{k_a : a \in \mathbb{D}\}$ is dense in $L_a^2(\mathbb{D})$.

Let $w \in \mathbb{C}_+$ and $\overline{w} = Ma, a \in \mathbb{D}$. Since $b_{\overline{w}} = Wk_a$ and W is an one-one, onto map from $L_a^2(\mathbb{D})$ onto $L_a^2(\mathbb{C}_+)$, hence $\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ span $L_a^2(\mathbb{C}_+)$. This can be verified as follows. Let $f \in L_a^2(\mathbb{C}_+)$. Then $f = Wg$, for some $g \in L_a^2(\mathbb{D})$. Now since $g = \lim_{n\to\infty} g_n$, where the functions g_n are linear combinations of certain normalized Bergman kernels $k_a, a \in \mathbb{D}$, hence $f = Wg = \lim_{n\to\infty} Wg_n$, where Wg_n is a linear combination of certain $b_{\overline{w}}, w \in \mathbb{C}_+$. Thus the set span$\{b_{\overline{w}} : w \in \mathbb{C}_+\}$ is dense in $L_a^2(\mathbb{C}_+)$. $\square$

For $\phi \in L^\infty(\mathbb{C}_+)$, we define $\widetilde{\phi}(w) = \langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$.

Theorem 4.3. *For $w \in \mathbb{C}_+, \phi \in L^\infty(\mathbb{C}_+)$, the following inequality hold:*

$$\widetilde{|\phi|^2}(w) - |\widetilde{\phi}(w)|^2 \leq ||\mathcal{H}_{\phi}|| + ||\mathcal{H}_{\overline{\phi}}||.$$

Proof. Notice that

$$
\begin{aligned}
\widetilde{|\phi|^2}(w) &= \langle \mathcal{T}_{|\phi|^2} b_{\overline{w}}, b_{\overline{w}} \rangle \\
&= \int_{\mathbb{C}_+} |\phi(z)|^2 \; |b_{\overline{w}}(z)|^2 d\mu(z) \\
&= \int_{\mathbb{C}_+} |(\phi \circ t_a)(z)|^2 \; |(b_{\overline{w}} \circ t_a)(z)|^2 |l_a(z)|^2 d\mu(z) \\
&= \left|\left| (\phi \circ t_a) \left(\frac{(-1)}{\sqrt{\pi}} M' \right) \right|\right|^2 . \qquad (4.1)
\end{aligned}
$$

Now

$$
\begin{aligned}
|\widetilde{\phi}(w)|^2 &= |\langle \mathcal{T}_{\phi} b_{\overline{w}}, b_{\overline{w}} \rangle|^2 \\
&= \left| \int_{\mathbb{C}_+} \phi(z) |b_{\overline{w}}(z)|^2 d\mu(z) \right|^2 \\
&= \left| \int_{\mathbb{C}_+} (\phi \circ t_a)(z) |(b_{\overline{w}} \circ t_a)(z)|^2 |l_a(z)|^2 d\mu(z) \right|^2 \\
&= \frac{1}{\pi^2} \left| \langle (\phi \circ t_a) M', M' \rangle \right|^2 . \qquad (4.2)
\end{aligned}
$$

Further,

$$
\begin{aligned}
||\mathcal{H}_{\phi}|| &= || \; (I - P_+)(\phi b_{\overline{w}}) \; ||^2 \\
&= \big|\big| \; (I - P_+) V_a [(\phi \circ t_a)(b_{\overline{w}} \circ t_a) l_a] \; \big|\big|^2 \\
&= \left(\frac{1}{\sqrt{\pi}} \right)^2 || \; (\phi \circ t_a) M' - P_+ \left[(\phi \circ t_a) M' \right] \; ||^2 \\
&= \frac{1}{\pi} || \; (\phi \circ t_a) M' - P_+ \left[(\phi \circ t_a) M' \right] \; ||^2 . \qquad (4.3)
\end{aligned}
$$

Thus from (4.1) and (4.2), it follows that $\widetilde{|\phi|^2}(w) - |\widetilde{\phi}(w)|^2$ equals

$$
\begin{aligned}
&\frac{1}{\pi}\,||(\phi\circ t_a)M'||^2 - \frac{1}{\pi^2}\big|\big\langle(\phi\circ t_a)M', M'\big\rangle\big|^2\\
&= \frac{1}{\pi}\left[\big|\big|(\phi\circ t_a)M' - P_+\big[(\phi\circ t_a)M'\big]\big|\big|^2 + \left|\left|P_+\big[(\phi\circ t_a)M'\big] - \frac{1}{\pi}\big\langle(\phi\circ t_a)M', M'\big\rangle M'\right|\right|^2\right]\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + \frac{1}{\pi}\left|\left|P_+\big[(\phi\circ t_a)M'\big] - \frac{1}{\pi}\big\langle(\phi\circ t_a)M', M'\big\rangle M'\right|\right|^2\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + ||P_+W(\phi\circ M\circ\phi_a) - \langle W(\phi\circ M\circ\phi_a), W1\rangle W1||^2\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + \frac{1}{\pi}\,\big|\big|W^{-1}\left((\overline{\phi}\circ t_a)M'\right) - PW^{-1}\left((\overline{\phi}\circ t_a)M'\right)\big|\big|^2\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + \frac{1}{\pi}\,\big|\big|W^{-1}\left((\overline{\phi}\circ t_a)M'\right) - W^{-1}P_+\left((\overline{\phi}\circ t_a)M'\right)\big|\big|^2\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + \frac{1}{\pi}\,\big|\big|(\overline{\phi}\circ t_a)M' - P_+\left((\overline{\phi}\circ t_a)M'\right)\big|\big|^2\\
&= ||\mathcal{H}_\phi b_{\overline{w}}||^2 + ||\mathcal{H}_{\overline{\phi}} b_{\overline{w}}||^2,
\end{aligned}
$$

since $W^{-1}P_+ = PW^{-1}$. □

5 Trace class operators

In this section, we prove that if A_1 is an operator in the trace class of $L^2_a(\mathbb{C}_+)$ then $tr(A_1) = \int_{\mathbb{C}_+}\widetilde{A}_1(w)d\nu(w)$ where $\widetilde{A}_1(w) = \langle A_1 b_{\overline{w}}, b_{\overline{w}}\rangle$. We also obtain the Schatten class characterization of positive Toeplitz operators.

Proposition 5.1. *Suppose $A_1 \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ is a positive operator on $L^2_a(\mathbb{C}_+)$ or A_1 is an operator in the trace class of $L^2_a(\mathbb{C}_+)$. Then*

$$tr(A_1) = \int_{\mathbb{C}_+}\widetilde{A}_1(w)d\nu(w)$$

where $\widetilde{A}_1(w) = \langle A_1 b_{\overline{w}}, b_{\overline{w}}\rangle$ and $d\nu(w) = |B(\overline{w}, w)|d\mu(w)$.

Proof. Since $A_1 \in \mathcal{L}(L_a^2(\mathbb{C}_+))$, hence $A_1 = WAW^{-1}$ for some positive operator $A \in \mathcal{L}(L_a^2(\mathbb{D}))$. Further A is in trace class in $\mathcal{L}(L_a^2(\mathbb{D}))$ if and only if A_1 is in trace class in $\mathcal{L}(L_a^2(\mathbb{C}_+))$. Notice that

$$\begin{aligned}
\int_{\mathbb{C}_+} \langle A_1 b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) &= \int_{\mathbb{C}_+} \langle A_1 b_{\overline{w}}, b_{\overline{w}} \rangle |B(\overline{w}, w)| d\mu(w) \\
&= \frac{1}{\pi} \int_{\mathbb{C}_+} \langle A_1 b_{\overline{w}}, b_{\overline{w}} \rangle \frac{|1+\overline{w}|^4}{(2(w+\overline{w}))^2} \frac{4}{|1+w|^4} d\mu(w) \\
&= \int_{\mathbb{D}} \langle A k_a, k_a \rangle \frac{dA(a)}{(1-|M\overline{w}|^2)^2} \text{ (where } w = M\overline{a}, a \in \mathbb{D}) \\
&= \int_{\mathbb{D}} \langle A k_a, k_a \rangle K(a,a) dA(a) \\
&= \int_{\mathbb{D}} \left\langle A \left(\sum_{n=1}^{\infty} e_n(a) \overline{e_n(a)} \right), K_a \right\rangle dA(a) \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \langle A e_n, K_a \rangle \overline{e_n(a)} dA(a) \\
&= \sum_{n=1}^{\infty} \langle A e_n, e_n \rangle = tr(A) = tr(A_1).
\end{aligned}$$

□

Proposition 5.2. *If ϕ is a nonnegative function on $\mathbb{C}_+$, then*

$$tr(\mathfrak{T}_\phi) = \int_{\mathbb{C}_+} \phi(w) d\nu(w).$$

Proof. By Proposition 5.1 and Fubini's theorem [15], we have

$$\begin{aligned}
tr(\mathfrak{T}_\phi) &= \int_{\mathbb{C}_+} \langle \mathfrak{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle |B(\overline{w}, w)| d\mu(w) \\
&= \int_{\mathbb{C}_+} |B(\overline{w}, w)| d\mu(w) \int_{\mathbb{C}_+} \phi(z) |b_{\overline{w}}(z)|^2 d\mu(z) \\
&= \int_{\mathbb{C}_+} d\mu(w) \int_{\mathbb{C}_+} \phi(z) |B(z,w)|^2 d\mu(z) \\
&= \int_{\mathbb{C}_+} \phi(z) d\mu(z) \int_{\mathbb{C}_+} |B(z,w)|^2 d\mu(w)
\end{aligned}$$

$$= \int_{\mathbb{C}_+} \phi(z) d\mu(z) \int_{\mathbb{C}_+} |B_{\overline{z}}(w)|^2 d\mu(w)$$
$$= \int_{\mathbb{C}_+} \phi(z) ||B_{\overline{z}}||^2 d\mu(z)$$
$$= \int_{\mathbb{C}_+} \phi(z) |B(\overline{z}, z)| d\mu(z).$$

The result follows. □

For $h > 1$, the generalized Kantorovich constant $K(h, p)$ is defined by

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p,$$

for any real number p and when there is no confusion, we write $K(h, p) = K(p)$.

Theorem 5.3. *Let A be a strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x and are equivalent:*

$$K(p)\langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq \langle Ax, x\rangle^p, \tag{5.1}$$

for any $p > 1$ or any $p < 0$ and

$$\langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq K(p)\langle Ax, x\rangle^p, \tag{5.2}$$

for any $p \in (0, 1]$.

Proof. For proof see [7]. □

The Kantorivch constant $K(p) \in (0, 1]$ for $p \in [0, 1], K(p)$ is symmetric with respect to $p = \frac{1}{2}$ and $K(p)$ is an increasing function of p for $p \geq \frac{1}{2}$ and $K(p)$ is a decreasing function of p for $p \leq \frac{1}{2}$, and $K(0) = K(1) = 1$. Further $K(p) \geq 1$ for $p \geq 1$ or $p \leq 0$ and $1 \geq K(p) \geq \frac{2h^{\frac{1}{4}}}{h^{\frac{1}{2}}+1}$ for $p \in [0, 1]$.

Proposition 5.4. *Let $\phi \in L^\infty(\mathbb{C}_+)$. Suppose $\mathcal{T}_\phi \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ is strictly positive satisfying $MI \geq \mathcal{T}_\phi \geq mI > 0$, where $M > m > 0$. The following hold:*

1. *If $0 < p < \infty$ and $\mathcal{T}_\phi \in S_p$ then $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$.*

2. If $0 < p \leq 1, \widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ *then* $\mathcal{T}_\phi \in S_p$.

3. Let $p \in [1, \infty)$ *be such that* $K(p) < \infty$. *If* $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ *then* $\mathcal{T}_\phi \in S_p$.

Proof. Suppose $p > 1$ and $\mathcal{T}_\phi \in S_p$. Then

$$\int_{\mathbb{C}_+} \langle \mathcal{T}_\phi^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) = \int_{\mathbb{C}_+} \langle |\mathcal{T}_\phi|^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty.$$

Hence by (5.1), $\displaystyle\int_{\mathbb{C}_+} \langle \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle^p d\nu(w) < \infty$. That is, $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$. Suppose $0 < p \leq 1$ and $\mathcal{T}_\phi \in S_p$. Then $\displaystyle\int_{\mathbb{C}_+} \langle \mathcal{T}_\phi^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) = \int_{\mathbb{C}_+} \langle |\mathcal{T}_\phi|^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty$. Hence from (5.2), it follows that $K(p) \int_{\mathbb{C}_+} \langle \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle^p d\nu(w) < \infty$. Since $K(p) \in (0, 1]$ for $p \in [0, 1]$, hence $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$.

Now assume $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$. Then if $0 < p \leq 1$ then by (5.2), we have $\displaystyle\int_{\mathbb{C}_+} \langle |\mathcal{T}_\phi|^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty$ and hence $\mathcal{T}_\phi \in S_p$. If $1 \leq p < \infty$, then by (5.1) and (5.2), if $K(p) < \infty$ and $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ then $\displaystyle\int_{\mathbb{C}_+} \langle |\mathcal{T}_\phi|^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty$ and $\mathcal{T}_\phi \in S_p$. □

6 Schatten class Toeplitz operator

In this section, we present conditions to describe Schatten class Toeplitz operators on the Bergman space $L_a^2(\mathbb{C}_+)$ of the right half plane.

Let $BT(\mathbb{D}) = \{f \in L^1(\mathbb{D}, dA) : ||f||_{BT(\mathbb{D})} = \sup_{a \in \mathbb{D}} \langle T_{|f|} k_a, k_a \rangle < \infty\}$. The space $L^\infty(\mathbb{D})$ is properly contained in $BT(\mathbb{D})$ (see [13]) and if $\phi \in BT(\mathbb{D})$ then T_ϕ is bounded on $L_a^2(\mathbb{D})$ and there is a constant C such that $||T_\phi|| \leq C||\phi||_{BT(\mathbb{D})}$.

Theorem 6.1. *Suppose* $1 \leq p < \infty$ *and* $d\nu(w) = |B(\overline{w}, w)| d\mu(w)$. *The following hold:*

(1) If $\mathcal{T}_\phi \in S_p$, *then* $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$.

(2) If $\phi \in L^p(\mathbb{C}_+, d\nu)$, *then* $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ *and* $\mathcal{T}_\phi \in S_p$.

Proof. Suppose $\mathcal{T}_\phi \in S_p$ and $w = M\overline{a}$. Then by Proposition 5.1, we have

$$\int_{\mathbb{C}_+} \langle |\mathcal{T}_\phi|^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty.$$

That is,

$$\int_{\mathbb{C}_+} \langle \left(\mathcal{T}_\phi^* \mathcal{T}_\phi\right)^{\frac{p}{2}} b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty.$$

If $2 \le p < \infty$, then

$$\int_{\mathbb{C}_+} \left\langle \mathcal{T}_\phi^* \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \right\rangle^{\frac{p}{2}} d\nu(w) \le \int_{\mathbb{C}_+} \left\langle \left(\mathcal{T}_\phi^* \mathcal{T}_\phi\right)^{\frac{p}{2}} b_{\overline{w}}, b_{\overline{w}} \right\rangle d\nu(w) < \infty.$$

Now since $||b_{\overline{w}}||_2 = 1$, we obtain

$$\begin{aligned}
\frac{1}{\pi} \left|\langle P_+ \left((\phi \circ t_a) M'\right), M' \rangle\right| &= \left|\left\langle P_+ \left((\phi \circ t_a) \left(\frac{(-1)}{\sqrt{\pi}} M'\right)\right), \left(\frac{(-1)}{\sqrt{\pi}} M'\right)\right\rangle\right| \\
&= \left|\langle P_+ \left((\phi \circ t_a) V_a b_{\overline{w}}\right), V_a b_{\overline{w}} \rangle\right| \\
&= \left|\left| P_+ \left((\phi \circ t_a) \left(\frac{(-1)}{\sqrt{\pi}} M'\right)\right)\right|\right| \\
&= \frac{1}{\sqrt{\pi}} \left|\left| P_+ \left((\phi \circ t_a) M'\right)\right|\right|. \qquad (6.1)
\end{aligned}$$

But from Proposition 2.3, it follows that

$$\begin{aligned}
\frac{1}{\pi} \left|\langle P_+((\phi \circ t_a) M'), M' \rangle\right| &= \frac{1}{\pi} \left|\langle \mathcal{T}_{\phi \circ t_a} M', M' \rangle\right| \\
&= \left|\left\langle \mathcal{T}_\phi b_{\overline{w}}, \left(\frac{-1}{\sqrt{\pi}}\right) V_a M' \right\rangle\right| \\
&= \left|\langle \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle\right|.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\pi^{\frac{p}{2}}} \int_{\mathbb{C}_+} \left|\left| P_+ \left((\phi \circ t_a) M'\right)\right|\right|^p d\nu(w) &= \int_{\mathbb{C}_+} \left|\left| P_+ \left((\phi \circ t_a) \left(\frac{(-1)}{\sqrt{\pi}} M'\right)\right)\right|\right|^p d\nu(w) \\
&= \int_{\mathbb{C}_+} \left|\left| P_+ \left((\phi \circ t_a) V_a b_{\overline{w}}\right)\right|\right|^p d\nu(w) \\
&= \int_{\mathbb{C}_+} \left\langle \mathcal{T}_\phi^* \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \right\rangle^{\frac{p}{2}} d\nu(w).
\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{\pi^p}\int_{\mathbb{C}_+}|\widetilde{\phi}(w)|^p d\nu(w) &= \frac{1}{\pi^p}\int_{\mathbb{C}_+}|\langle \mathfrak{T}_\phi b_{\overline{w}}, b_{\overline{w}}\rangle|^p\, d\nu(w)\\ &= \frac{1}{\pi^p}\int_{\mathbb{C}_+}|\langle P_+\left((\phi\circ t_a)M'\right), M'\rangle|^p\, d\nu(w)\\ &\le \frac{1}{\pi^{\frac{p}{2}}}\int_{\mathbb{C}_+}||P_+\left((\phi\circ t_a)M'\right)||^p\, d\nu(w)\\ &= \int_{\mathbb{C}_+}\left\langle \mathfrak{T}_\phi^*\mathfrak{T}_\phi b_{\overline{w}}, b_{\overline{w}}\right\rangle^{\frac{p}{2}} d\nu(w) < \infty.\end{aligned}$$

That is, $\int_{\mathbb{C}_+}|\widetilde{\phi}(w)|^p d\nu(w) < \infty$ and $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$, where $\widetilde{\phi}(w) = \widetilde{\mathfrak{T}_\phi}(w) = \langle \mathfrak{T}_\phi b_{\overline{w}}, b_{\overline{w}}\rangle$. Suppose $1 \le p < 2$. Then by Heinz inequality [10], it follows from (6.1) that

$$\begin{aligned}\infty &> \int_{\mathbb{C}_+}\langle |\mathfrak{T}_\phi|^p\, b_{\overline{w}}, b_{\overline{w}}\rangle\, d\nu(w)\\ &\ge \int_{\mathbb{C}_+}\frac{|\langle \mathfrak{T}_\phi b_{\overline{w}}, b_{\overline{w}}\rangle|^2}{\left\langle |\mathfrak{T}_\phi^*|^{2(1-\frac{p}{2})} b_{\overline{w}}, b_{\overline{w}}\right\rangle} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{|\widetilde{\phi}(w)|^2}{\left(\frac{1}{\sqrt{\pi}}\left|\left|P_+\left((\overline{\phi}\circ t_a)M'\right)\right|\right|\right)^{2-p}} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{|\widetilde{\phi}(w)|^2}{\frac{1}{\pi^{\frac{2-p}{2}}}\left|\left|P_+\left((\overline{\phi}\circ t_a)M'\right)\right|\right|^{2-p}} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{|\widetilde{\phi}(w)|^2}{\frac{1}{\pi^{\frac{2-p}{2}}}\left|\left|\mathfrak{T}_{\overline{\phi}\circ t_a}M'\right|\right|^{2-p}} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{|\widetilde{\phi}(w)|^2}{\left|\left|V_a\mathfrak{T}_{\overline{\phi}}V_a\left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right|\right|^{2-p}} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{|\widetilde{\phi}(w)|^2}{||\mathfrak{T}_{\overline{\phi}}b_{\overline{w}}||^{2-p}} d\nu(w)\\ &= \int_{\mathbb{C}_+}\frac{||\mathfrak{T}_{\overline{\phi}}b_{\overline{w}}||^p\ |\widetilde{\phi}(w)|^2}{||\mathfrak{T}_{\overline{\phi}}b_{\overline{w}}||^2} d\nu(w)\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\mathbb{C}_+} \frac{|\widetilde{\phi}(w)|^2}{||W^{-1}\mathfrak{T}_{\overline{\phi}}Wk_a||^2} \left(\frac{1}{\sqrt{\pi}} ||P_+ \left((\overline{\phi} \circ t_a)M' \right)|| \right)^p d\nu(w) \\
&\geq \int_{\mathbb{C}_+} \frac{|\widetilde{\phi}(w)|^2}{||T_{\overline{\phi}\circ M}k_a||^2} \left(\frac{1}{\pi} \left| \left\langle P_+ \left((\overline{\phi} \circ t_a)M' \right), M' \right\rangle \right| \right)^p d\nu(w) \\
&= \int_{\mathbb{C}_+} \frac{|\widetilde{\phi}(w)|^2}{||T_{\overline{\phi}\circ M}k_a||^2} \left(\frac{1}{\pi} \left| \left\langle P_+ \left((\phi \circ t_a)M' \right), M' \right\rangle \right| \right)^p d\nu(w) \\
&\geq \int_{\mathbb{C}_+} \frac{1}{C^2||\phi \circ M||^2_{BT(\mathbb{D})}} |\widetilde{\phi}(w)|^2 |\widetilde{\phi}(w)|^p d\nu(w).
\end{aligned}
$$

Since

$$
\begin{aligned}
\langle |\mathfrak{T}^*_{\phi}|^{2-p} b_{\overline{w}}, b_{\overline{w}} \rangle &= \langle |\mathfrak{T}^*_{\phi}|^{2\cdot\frac{(2-p)}{2}} b_{\overline{w}}, b_{\overline{w}} \rangle \\
&\leq \langle |\mathfrak{T}^*_{\phi}|^2 b_{\overline{w}}, b_{\overline{w}} \rangle^{\frac{2-p}{2}} \\
&= \langle \mathfrak{T}_{\phi}\mathfrak{T}^*_{\phi} b_{\overline{w}}, b_{\overline{w}} \rangle^{\frac{2-p}{2}} \\
&= ||\mathfrak{T}^*_{\phi} b_{\overline{w}}||^{2-p} \\
&= \left|\left| V_a \mathfrak{T}_{\overline{\phi}} V_a \left(\frac{(-1)}{\sqrt{\pi}} M' \right) \right|\right|^{2-p} \\
&= \frac{1}{\pi^{\frac{2-p}{2}}} ||\mathfrak{T}_{\overline{\phi}\circ t_a} M'||^{2-p} \\
&= \frac{1}{\pi^{\frac{2-p}{2}}} ||P_+((\overline{\phi} \circ t_a)M')||^{2-p} \\
&= \left(\frac{1}{\sqrt{\pi}} \left|\left| P_+ \left((\overline{\phi} \circ t_a)M' \right) \right|\right| \right)^{2-p}.
\end{aligned}
$$

Hence $\int_{\mathbb{C}_+} |\widetilde{\phi}(w)|^{p+2} d\nu(w) < \infty$ and therefore $\int_{\mathbb{C}_+} |\widetilde{\phi}(w)|^p d\nu(w) < \infty$. Thus $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$. Now suppose $\phi \in L^1(\mathbb{C}_+, d\nu)$. Then the change of the order of integration

$$\begin{aligned}
\int_{\mathbb{C}_+} |\widetilde{\phi}(w)| d\nu(w) &= \int_{\mathbb{C}_+} |\widetilde{\phi}(w)| \; |B(\overline{w}, w)| \; d\mu(w) \\
&\leq \int_{\mathbb{C}_+} \left(\int_{\mathbb{C}_+} |\phi(z)| \; |b_{\overline{w}}(z)|^2 d\mu(z) \right) |B(\overline{w}, w)| \; d\mu(w) \\
&= \int_{\mathbb{C}_+} |\phi(z)| \int_{\mathbb{C}_+} |B_{\overline{w}}(z)|^2 d\mu(w) \; d\mu(z) \\
&= \int_{\mathbb{C}_+} |\phi(z)| \langle B_{\overline{z}}, B_{\overline{z}} \rangle \; d\mu(z) \\
&= \int_{\mathbb{C}_+} |\phi(z)| B(\overline{z}, z) \; d\mu(z),
\end{aligned}$$

is justified by the positivity of the integrand. Hence $\widetilde{\phi} \in L^1(\mathbb{C}_+, d\nu)$. Similarly, if $\phi \in L^\infty(\mathbb{C}_+)$ then $\widetilde{\phi} \in L^\infty(\mathbb{C}_+)$ as $|\widetilde{\phi}(w)| = |\langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle| \leq ||\phi b_{\overline{w}}||_2 ||b_{\overline{w}}||_2 \leq ||\phi||_\infty ||b_{\overline{w}}||_2^2 = ||\phi||_\infty$. By Marcinkiewicz interpolation theorem [19], it follows that if $\phi \in L^p(\mathbb{C}_+, d\nu)$ then $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$ for $1 \leq p \leq \infty$. Now suppose $\phi \in L^p(\mathbb{C}_+, d\nu), 1 \leq p \leq \infty$. We will prove $\mathfrak{T}_\phi \in S_p$. The case $p = +\infty$ is trivial. By interpolation we need only to prove the result for $p = 1$. Suppose $\phi \in L^1(\mathbb{C}_+, d\nu)$. The vectors

$$\begin{aligned}
\epsilon_n(z) = (We_n)(z) &= \frac{(-1)}{\sqrt{\pi}} (e_n \circ M)(z) M'(z) \\
&= \frac{(-1)}{\sqrt{\pi}} e_n(Mz) M'(z) \\
&= \frac{(-1)}{\sqrt{\pi}} \sqrt{n+1} \left(\frac{1-z}{1+z} \right)^n \left(\frac{(-2)}{(1+z)^2} \right) \\
&= \frac{2\sqrt{n+1}}{\sqrt{\pi}} \left(\frac{1-z}{1+z} \right)^n \frac{1}{(1+z)^2}, \; n = 1, 2, 3, \cdots
\end{aligned}$$

forms an orthonormal basis for $L^2_a(\mathbb{C}_+)$. Now $\langle \mathfrak{T}_\phi \epsilon_n, \epsilon_n \rangle = \displaystyle\int_{\mathbb{C}_+} |\epsilon_n(z)|^2 \phi(z) \, d\mu(z)$ and

$$\begin{aligned}\sum_{n=0}^{\infty}|\langle \mathfrak{T}_\phi \epsilon_n, \epsilon_n\rangle| &\leq \int_{\mathbb{C}_+}\sum_{n=0}^{\infty}|\epsilon_n(z)|^2|\phi(z)|\ d\mu(z)\\ &= \int_{\mathbb{C}_+}\frac{1}{\pi}\left(\sum_{n=0}^{\infty}|e_n(Mz)|^2|M'(z)|^2\right)|\phi(z)|\ d\mu(z)\\ &= \int_{\mathbb{D}}|B(\overline{w},w)|\ \frac{|K(a,a)|}{|B(\overline{w},w)|}\ |(\phi\circ M)(a)|\ dA(a)\\ &= \int_{\mathbb{D}}|(\phi\circ M)(a)|\ |B(\overline{w},w)|\ \pi\ |M'(a)|^2\ dA(a)\end{aligned}$$

$$\begin{aligned}&= \int_{\mathbb{D}}|(\phi\circ M)(a)|\ |B(Ma,M\overline{a})|\ d\mu(Ma)\\ &= \int_{\mathbb{C}_+}|\phi(w)|\ d\nu(w),\end{aligned}$$

since

$$\begin{aligned}\frac{|K(a,a)|}{|B(\overline{w},w)|} &= \frac{1}{(1-|a|^2)^2}\frac{4\pi\ (1-|a|^2)^2}{|1+a|^4}\\ &= \frac{4\pi}{|1+a|^4}\\ &= \pi\left|\frac{(-2)}{(1+a)^2}\right|^2 = \pi|M'(a)|^2.\end{aligned}$$

Thus $\mathfrak{T}_\phi \in S_1$ and $||\mathfrak{T}_\phi||_{S_1} \leq \int_{\mathbb{C}_+}|\phi(w)|d\nu(w)$. □

Define $(Bf)(z) = \int_{\mathbb{D}} f(z_1)|k_z(z_1)|^2 dA(z_1), z\in\mathbb{D}, f\in L^2(\mathbb{D},dA)$.

Proposition 6.2. *Suppose ϕ is a nonnegative function on $\mathbb{C}_+, 1\leq p<\infty$, then the following are equivalent:*

(1) $\mathfrak{T}_\phi$ is in Schatten class S_p;

(2) $\widetilde{\phi}(z)$ is in $L^p(\mathbb{C}_+, d\nu)$;

Proof. Suppose $1 \leq p < \infty$ and $\mathcal{T}_\phi \in S_p$. Then $\mathcal{T}_\phi^p \in S_1$. Since $\mathcal{T}_\phi \geq 0$, by Proposition 5.1, $tr(\mathcal{T}_\phi^p) = \int_{\mathbb{C}_+} \langle \mathcal{T}_\phi^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty$. By Theorem 5.3, $\int_{\mathbb{C}_+} [\widetilde{\phi}(w)]^p d\nu(w) = \int_{\mathbb{C}_+} [\langle \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle]^p \, d\nu(w) \leq \int_{\mathbb{C}_+} \langle \mathcal{T}_\phi^p b_{\overline{w}}, b_{\overline{w}} \rangle d\nu(w) < \infty$. Hence $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$. To prove the converse, suppose $\widetilde{\phi} \in L^p(\mathbb{C}_+, d\nu)$. Then since $\widetilde{\phi}(w) = \langle \phi b_{\overline{w}}, b_{\overline{w}} \rangle = \langle \mathcal{T}_\phi b_{\overline{w}}, b_{\overline{w}} \rangle = \langle W T_{\phi \circ M} W^{-1} W k_a, W k_a \rangle = \langle T_{\phi \circ M} k_a, k_a \rangle = B(\phi \circ M)(a)$. Hence $B(\phi \circ M) \in L^p(\mathbb{D}, d\lambda)$ where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}, z \in \mathbb{D}$. From [19], it follows that $T_{\phi \circ M} \in (S_p, \mathcal{L}(L_a^2(\mathbb{D})))$. By Lemma 4.1, it follows that $\mathcal{T}_\phi = W T_{\phi \circ M} W^{-1} \in (S_p, \mathcal{L}(L_a^2(\mathbb{C}_+)))$. □

7 Bounded linear operators on $L_a^2(\mathbb{C}_+)$

In this section, we find conditions on $C \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ such that $C \in S_p$, the Schatten p-class, $1 \leq p < \infty$ by comparing with positive Toeplitz operators defined the Bergman space $L_a^2(\mathbb{C}_+)$ and applications of the result are also obtained.

Theorem 7.1. *Let $\phi \in L^p(\mathbb{C}_+, d\nu), \psi \in L^q(\mathbb{C}_+, d\nu)$, where $1 \leq p, q < \infty$. Let $C \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ be such that*

$$|\langle C B_{\overline{v}}, B_{\overline{w}} \rangle|^2 \leq \langle \mathcal{T}_{|\phi|} B_{\overline{v}}, B_{\overline{v}} \rangle \langle \mathcal{T}_{|\psi|} B_{\overline{w}}, B_{\overline{w}} \rangle \tag{7.1}$$

for all $\overline{v}, \overline{w} \in \mathbb{C}_+$. Then $C \in S_{2r}$ and $||C||_{2r}^2 \leq ||\mathcal{T}_{|\phi|}||_p \, ||\mathcal{T}_{|\psi|}||_q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof. First we show that (7.1) implies

$$|\langle Cf, g \rangle|^2 \leq \langle \mathcal{T}_{|\phi|} f, f \rangle \langle \mathcal{T}_{|\psi|} g, g \rangle$$

for all $f, g \in L_a^2(\mathbb{C}_+)$. Let $f = \sum_{j=1}^{n} c_j B_{\overline{v}_j}$ where c_j are constants, $\overline{v}_j \in \mathbb{C}_+$ for $j = 1, 2, \cdots, n$ and $g = \sum_{i=1}^{m} d_i B_{\overline{w}_i}$ where d_i are constants, $\overline{w}_i \in \mathbb{C}_+$ for

$i = 1, 2, \cdots, m$. Then

$$\begin{aligned}
|\langle Cf, g\rangle| &= \left|\left\langle C\left(\sum_{j=1}^{n} c_j B_{\overline{v}_j}\right), \sum_{i=1}^{m} d_i B_{\overline{w}_i}\right\rangle\right| \\
&= \left|\sum_{i=1,j=1}^{m,n} c_j \overline{d}_i \left\langle CB_{\overline{v}_j}, B_{\overline{w}_i}\right\rangle\right| \\
&\leq \sum_{i=1,j=1}^{m,n} |c_j|\ |\overline{d}_i|\ |\langle CB_{\overline{v}_j}, B_{\overline{w}_i}\rangle| \\
&\leq \sum_{i=1,j=1}^{m,n} |c_j|\ |\overline{d}_i| \langle \mathfrak{T}_{|\phi|} B_{\overline{v}_j}, B_{\overline{v}_j}\rangle^{\frac{1}{2}} \langle \mathfrak{T}_{|\psi|} B_{\overline{w}_i}, B_{\overline{w}_i}\rangle^{\frac{1}{2}} \\
&= \left\langle \mathfrak{T}_{|\phi|}\left(\sum_{j=1}^{n} c_j B_{\overline{v}_j}\right), \sum_{j=1}^{n} c_j B_{\overline{v}_j}\right\rangle^{\frac{1}{2}} \left\langle \mathfrak{T}_{|\psi|}\left(\sum_{i=1}^{m} d_i B_{\overline{w}_i}\right), \sum_{i=1}^{m} d_i B_{\overline{w}_i}\right\rangle^{\frac{1}{2}} \\
&= \langle \mathfrak{T}_{|\phi|} f, f\rangle^{\frac{1}{2}} \langle \mathfrak{T}_{|\psi|} g, g\rangle^{\frac{1}{2}}.
\end{aligned}$$

From Lemma 4.2, it follows that the set of vectors $\left\{\sum c_j B_{\overline{w}_j}, \overline{w}_j \in \mathbb{C}_+, j = 1, 2, \cdots, n\right\}$ is dense in $L^2_a(\mathbb{C}_+)$. Hence

$$|\langle Cf, g\rangle|^2 \leq \langle \mathfrak{T}_{|\phi|} f, f\rangle \langle \mathfrak{T}_{|\psi|} g, g\rangle$$

for all $f, g \in L^2_a(\mathbb{C}_+)$. Since $\phi \in L^p(\mathbb{C}_+, d\nu)$, it follows from Theorem 6.1, that $\mathfrak{T}_{|\phi|} \in S_p$ and $||\mathfrak{T}_{|\phi|}||_p = \left(\text{trace}\mathfrak{T}^p_{|\phi|}\right)^{\frac{1}{p}} < \infty$. Similarly $\psi \in L^q(\mathbb{C}_+, d\nu)$, implies that $||\mathfrak{T}_{|\psi|}||_q = \left(\text{trace}\mathfrak{T}^q_{|\psi|}\right)^{\frac{1}{q}} < \infty$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{\xi_n\}_{n=0}^{\infty}$ be two orthonormal sequences in $L^2_a(\mathbb{C}_+)$. Then using Holder's inequality, we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} |\langle Cu_n, \xi_n\rangle|^{2r} &\leq \sum_{n=0}^{\infty} \langle \mathfrak{T}_{|\phi|} u_n, u_n\rangle^r \langle \mathfrak{T}_{|\psi|} \xi_n, \xi_n\rangle^r \\
&\leq \left(\sum_{n=0}^{\infty} \langle \mathfrak{T}_{|\phi|} u_n, u_n\rangle^p\right)^{\frac{r}{p}} \left(\sum_{n=0}^{\infty} \langle \mathfrak{T}_{|\psi|} \xi_n, \xi_n\rangle^q\right)^{\frac{r}{q}} \\
&\leq \left(\sum_{n=0}^{\infty} \langle \mathfrak{T}^p_{|\phi|} u_n, u_n\rangle\right)^{\frac{r}{p}} \left(\sum_{n=0}^{\infty} \langle \mathfrak{T}^q_{|\psi|} \xi_n, \xi_n\rangle\right)^{\frac{r}{q}} \\
&\leq \left(\text{trace}\mathfrak{T}^p_{|\phi|}\right)^{\frac{r}{p}} \left(\text{trace}\mathfrak{T}^q_{|\psi|}\right)^{\frac{r}{q}} \\
&= ||\mathfrak{T}_{|\phi|}||_p^r\ ||\mathfrak{T}_{|\psi|}||_q^r \quad \text{if } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
\end{aligned}$$

Thus $||C||_{2r} \leq ||\mathfrak{T}_{|\phi|}||_p^{\frac{1}{2}} ||\mathfrak{T}_{|\psi|}||_q^{\frac{1}{2}}$. $\square$

Corollary 7.2. *If $\phi, \psi \in L^p(\mathbb{C}_+, d\nu)$ and $C \in L(L_a^2(\mathbb{C}_+))$ is such that*

$$|\langle CB_{\overline{v}}, B_{\overline{w}}\rangle|^2 \leq \langle \mathfrak{T}_{|\phi|}B_{\overline{v}}, B_{\overline{v}}\rangle\langle \mathfrak{T}_{|\psi|}B_{\overline{w}}, B_{\overline{w}}\rangle$$

for all $\overline{v}, \overline{w} \in \mathbb{C}_+$ then $||C||_p^2 \leq ||\mathfrak{T}_{|\phi|}||_p \; ||\mathfrak{T}_{|\psi|}||_p$.

Proof. The proof follows from Theorem 7.1 if we assume $p = q$. $\square$

Corollary 7.3. *If S and T are two positive operators in $\mathcal{L}(L_a^2(\mathbb{C}_+))$ and $S \in S_p, T \in S_q, 1 \leq p, q < \infty$ and $C \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ is such that*

$$|\langle CB_{\overline{v}}, B_{\overline{w}}\rangle|^2 \leq \langle SB_{\overline{v}}, B_{\overline{v}}\rangle\langle TB_{\overline{w}}, B_{\overline{w}}\rangle$$

for all $\overline{v}, \overline{w} \in \mathbb{C}_+$. Then $||C||_{2r}^2 \leq ||S||_p||T||_q$ if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $p = q$, then $||C||_p^2 \leq ||S||_p||T||_p$.

Proof. Proceeding similarly as in Theorem 7.1 and Corollary 7.2 by replacing $\mathfrak{T}_{|\phi|}$ by S and $\mathfrak{T}_{|\psi|}$ by T, the corollary follows. $\square$

Corollary 7.4. *If $S, T \in \mathcal{L}(L_a^2(\mathbb{C}_+)), 0 \leq S \in S_p, 1 \leq p < \infty$ and $|\langle Cu_n, \xi_n\rangle|^2 \leq \langle Su_n, u_n\rangle\langle T\xi_n, \xi_n\rangle$, then $||C||_{2p}^2 \leq ||S||_p||T||$.*

Proof. Let $\{u_n\}_{n=0}^{\infty}$ and $\{\xi_n\}_{n=0}^{\infty}$ be two orthonormal bases for $L_a^2(\mathbb{C}_+)$, then

$$\begin{aligned} |\langle Cu_n, \xi_n\rangle|^2 &\leq \langle Su_n, u_n\rangle\langle T\xi_n, \xi_n\rangle \\ &\leq \langle Su_n, u_n\rangle||T||. \end{aligned}$$

Then $|\langle Cu_n, \xi_n\rangle|^{2p} \leq ||T||^p\langle Su_n, u_n\rangle^p$. Hence

$$\sum_{n=0}^{\infty} |\langle Cu_n, \xi_n\rangle|^{2p} \leq ||T||^p \sum_{n=0}^{\infty} \langle Su_n, u_n\rangle^p$$

and $||C||_{2p}^2 \leq ||T|| \; ||S||_p$. $\square$

By Theorem 6.1, if $\phi \in L^p(\mathbb{C}_+, d\nu)$ then $\mathfrak{T}_\phi \in S_p$. Hence it follows from [19], $|\mathfrak{T}_\phi| \in S_p$. Thus if $C, T \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ are such that $|\langle CB_{\overline{v}}, B_{\overline{w}}\rangle|^2 \leq \langle |\mathfrak{T}_\phi|B_{\overline{v}}, B_{\overline{w}}\rangle\langle TB_{\overline{v}}, B_{\overline{w}}\rangle$ for all $\overline{v}, \overline{w} \in \mathbb{C}_+$ then $C \in S_{2p}$ and $||C||_{2p}^2 \leq ||T|| \; || \, |\mathfrak{T}_\phi| \, ||_p$.

Corollary 7.5. *Let $\phi \in L^p(\mathbb{C}_+, d\nu), 1 < p < \infty$ and $\phi = \phi^+$ where $\phi^+(w) = \overline{\phi(\overline{w})}$. Then there exists an operator $S \in \mathcal{L}(L_a^2(\mathbb{C}_+))$ such that $\mathfrak{T}_{|\phi|}S = S\mathfrak{T}_{|\phi|}$ and $||\mathfrak{T}_{|\phi|}S||_p \leq r(S)||\mathfrak{T}_{|\phi|}||_p$ where $r(S)$ is the spectral radius of S.*

Proof. Since $\phi \in L^p(\mathbb{C}_+, d\nu)$ and $\phi^+ = \phi$, hence $\mathcal{T}_{|\phi|}$ and $\mathcal{S}_\phi$ are self-adjoint operators, $\mathcal{T}_{|\phi|} \in S_p$ and $\mathcal{S}_\phi \in S_p$. Let $\mathcal{U}$ be the group of unitary operators on $L^2_a(\mathbb{C}_+)$. Let $\mathcal{U}_A = \{UAU^* : U \in \mathcal{U}\}$, the unitary orbit of an operator $A \in \mathcal{L}(L^2_a(\mathbb{C}_+))$.

Define $f : S_p \longrightarrow \mathbb{R}$ as $f(X) = ||\mathcal{T}_{|\phi|} - X||_p$. Then f attains its minimum at some $S \in S_p$ on $\mathcal{U}_{\mathcal{S}_\phi} = \{U\mathcal{S}_\phi U^* : U \in \mathcal{U}\}$ and $\mathcal{T}_{|\phi|}S = S\mathcal{T}_{|\phi|}$. This follows from [1]. The operator S is also self-adjoint. To prove the corollary we have to show that for any two orthonormal sequences $\{u_n\}_{n=0}^\infty$ and $\{\xi_n\}_{n=0}^\infty$ in $L^2_a(\mathbb{C}_+)$,

$$\sum_{n=0}^{\infty} |\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^p \leq r(S)^p ||\mathcal{T}_{|\phi|}||_p^p.$$

Since $\mathcal{T}_{|\phi|}S = S\mathcal{T}_{|\phi|}$ and $S = S^*$, it follows from Reid's inequality [12], that

$$\begin{aligned} |\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^2 &= |\langle \mathcal{T}_{|\phi|}(Su_n), \xi_n\rangle|^2 \\ &\leq \langle \mathcal{T}_{|\phi|}(Su_n), Su_n\rangle \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle \\ &= \langle S^*\mathcal{T}_{|\phi|} S u_n, u_n\rangle \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle \\ &= \langle \mathcal{T}_{|\phi|} S^2 u_n, u_n\rangle \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle. \end{aligned} \tag{7.2}$$

Now from (7.2), it follows that

$$\begin{aligned} |\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^{2^{m+1}} &= \left(|\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^{2^m}\right)^2 \\ &= \left(\left(|\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^2\right)^{2^{m-1}}\right)^2 \\ &\leq \left(\left(\langle \mathcal{T}_{|\phi|} S^2 u_n, u_n\rangle \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle\right)^{2^{m-1}}\right)^2 \\ &= \left(\left(|\langle \mathcal{T}_{|\phi|} S^2 u_n, u_n\rangle|\right)^{2^{m-1}}\right)^2 \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m} \\ &= \left(\left(|\langle \mathcal{T}_{|\phi|} S^2 u_n, u_n\rangle|^2\right)^{2^{m-2}}\right)^2 \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m} \\ &\leq \left(\left(|\langle \mathcal{T}_{|\phi|} S^{2^2} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle|\right)^{2^{m-2}}\right)^2 \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m} \\ &= \left(|\langle \mathcal{T}_{|\phi|} S^{2^2} u_n, u_n\rangle|^{2^{m-2}}\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-1}} \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m} \\ &= \left(\left(|\langle \mathcal{T}_{|\phi|} S^{2^2} u_n, u_n\rangle|^2\right)^{2^{m-3}}\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-1}} \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m} \end{aligned}$$

$$\leq \left(\left(|\langle \mathcal{T}_{|\phi|} S^{2^3} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle|\right)^{2^{m-3}}\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-1}} \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m}$$

$$= \left(|\langle \mathcal{T}_{|\phi|} S^{2^3} u_n, u_n\rangle|^{2^{m-3}}\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-2}} \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-1}} \langle \mathcal{T}_{|\phi|}\xi_n, \xi_n\rangle^{2^m}.$$

Repeating this process, we obtain

$$\begin{aligned}
\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^{2^{m+1}} &\leq \left(|\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle|\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-(m-1)}+\cdots+2^{m-1}} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \left(|\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle|\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{1+\cdots+2^{m-1}}} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \left(|\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle|\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2(1+\cdots+2^{m-2})} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \left(|\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle|\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{\frac{2(2^{m-1}-1)}{2-1}} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \left(|\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle|\right)^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2(2^{m-1}-1)} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&\leq |\langle \mathcal{T}_{|\phi|} S^{2^m} u_n, S^{2^m} u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^m-2} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \langle S^{*^{2^m}} \mathcal{T}_{|\phi|} S^{2^m} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^m-1} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m} \\
&= \langle \mathcal{T}_{|\phi|} S^{2^{m+1}} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^m-1} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^m}.
\end{aligned}$$

Thus

$$|\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^{2^m} \leq ||\mathcal{T}_{|\phi|}||\ ||S^{2^m}||\ ||u_n||^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{2^{m-1}-1} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{2^{m-1}}$$

and

$$|\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle| \leq ||\mathcal{T}_{|\phi|}||^{\frac{1}{2^m}}\ ||S^{2^m}||^{\frac{1}{2^m}}\ ||u_n||^{\frac{2}{2^m}} \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle^{\frac{1}{2}-\frac{1}{2^m}} \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle^{\frac{1}{2}}.$$

Letting $m \longrightarrow \infty$, we obtain

$$|\langle \mathcal{T}_{|\phi|} S u_n, \xi_n\rangle|^2 \leq [r(S)]^2 \langle \mathcal{T}_{|\phi|} u_n, u_n\rangle \langle \mathcal{T}_{|\phi|} \xi_n, \xi_n\rangle.$$

Hence proceeding as in Theorem 7.1 and Corollary 7.2, we can show that $||\mathcal{T}_{|\phi|} S||_p \leq r(S) ||\mathcal{T}_{|\phi|}||_p$. □

8 Little Hankel operators

In this section, we show that the Schatten class properties of the little Hankel operator $\hbar_{\overline{f}} = J\mathcal{S}_{\overline{f}}, f \in L^2(\mathbb{C}_+, d\mu)$ depends only on the anti-analytic part of the symbol and establish that for $2 \leq p < \infty$, the little Hankel operator $\mathcal{S}_{\overline{\phi}} \in S_p$ if and only if $\mathcal{V}_1\phi \in L^p(\mathbb{C}_+, d\nu)$ where $d\nu(w) = |B(\overline{w}, w)| d\mu(w)$.

Let $H^\infty(\mathbb{C}_+)$ be the space of bounded analytic functions on $\mathbb{C}_+$. It is not difficult to verify that $H^\infty(\mathbb{C}_+) = WH^\infty(\mathbb{D})$.

Proposition 8.1. *If $f \in L^2(\mathbb{C}_+, d\mu)$, then $\hbar_{\overline{f}} = \hbar_{\overline{P_+ f}}$ in the sense that $\hbar_{\overline{f}} g = \hbar_{\overline{P_+ f}} g$ for all $g \in H^\infty(\mathbb{C}_+)$(which is dense in $L^2_a(\mathbb{C}_+)$) .*

Proof. Let $h \in L^2_a(\mathbb{C}_+)$ and $g \in H^\infty(\mathbb{C}_+)$. Then

$$\begin{aligned}\langle \hbar_{\overline{f}} g, \overline{h}\rangle &= \langle \overline{P}_+(\overline{f}g), \overline{h}\rangle \\ &= \langle \overline{P_+ f} g, \overline{h}\rangle \\ &= \langle \overline{P_+ f} g, \overline{P}_+ \overline{h}\rangle \\ &= \langle \hbar_{\overline{P_+ f}} g, \overline{h}\rangle .\end{aligned}$$

Hence $\hbar_{\overline{f}} g = \hbar_{\overline{P_+ f}} g$ for all $g \in H^\infty(\mathbb{C}_+)$. $\square$

Thus from Proposition 8.1, it follows that for $f \in L^2(\mathbb{C}_+, d\mu), \mathcal{S}_{\overline{f}} = \mathcal{S}_{\overline{P_+ f}}$.

For $f \in L^2(\mathbb{C}_+, d\mu)$, define $(\mathcal{V}_1 f)(w) = 3\langle \overline{b}_{\overline{w}}, \hbar_{\overline{f}} b_{\overline{w}}\rangle$. It is not so difficult to see that $(i)\mathcal{V}_1 P_+ = \mathcal{V}_1 (ii) P_+ \mathcal{V}_1 = P_+$ and $(iii)\mathcal{V}_1^2 = \mathcal{V}_1$. This can be verified as follows. From Proposition 8.1, we obtain $\mathcal{V}_1 P_+ f = 3\langle \overline{b}_{\overline{w}}, \hbar_{\overline{P_+ f}} b_{\overline{w}}\rangle = 3\langle \overline{b}_{\overline{w}}, \hbar_{\overline{f}} b_{\overline{w}}\rangle = \mathcal{V}_1 f$ for $f \in L^2(\mathbb{C}_+, d\mu)$. Now let $f, g \in L^2(\mathbb{C}_+, d\mu)$ and $g = g_1 + g_2$ where $g_1 \in L^2_a(\mathbb{C}_+)$ and $g_2 \in (L^2_a(\mathbb{C}_+))^\perp$. Then

$$\begin{aligned}\langle P_+ \mathcal{V}_1 f, g\rangle &= \langle \mathcal{V}_1 f, P_+ g\rangle \\ &= \langle \mathcal{V}_1 f, g_1\rangle \\ &= \pi \int_{\mathbb{D}} [V(f \circ M)](z)\overline{(g_1 \circ M)(z)} |M'(z)|^2 \, dA(z),\end{aligned}$$

where $(Vh)(z) = 3(1-|z|^2)^2 \displaystyle\int_{\mathbb{D}} \frac{h(u)}{(1-z\overline{u})^4} dA(u)$ for $h \in L^2(\mathbb{D}, dA)$. Under the complex integral pairing with respect to dA, we have $V = P_2^*$ where $P_2 h(z) = 3 \displaystyle\int_{\mathbb{D}} \frac{(1-|u|^2)^2}{(1-z\overline{u})^4} h(u) dA(u)$ is a projection from $L^1(\mathbb{D}, dA)$ onto $L^1_a(\mathbb{D})$. From Fubini's theorem [15] and the fact that both P and P_2 reproduce analytic functions it follows that $PV = P$ where P is the Bergman projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Thus for $f, g \in L^2(\mathbb{C}_+, d\mu)$,

$$\langle P_+\mathcal{V}_1 f, g\rangle = \pi\int_{\mathbb{D}}[V(f\circ M)](z)\overline{(g_1\circ M)(z)}\ |M'(z)|^2\ dA(z)$$

$$= \pi\int_{\mathbb{D}} V[(f\circ M)M'](z)\overline{(g_1\circ M)(z)M'(z)}\ dA(z)$$

$$= \int_{\mathbb{D}} V[(-1)\sqrt{\pi}(f\circ M)M'](z)\overline{(-1)\sqrt{\pi}(g_1\circ M)(z)M'(z)}\ dA(z)$$

$$= \int_{\mathbb{D}} V(W^{-1}f)(z)\overline{(W^{-1}g_1)(z)}\ dA(z)$$

$$= \langle WPW^{-1}f, g_1\rangle$$

$$= \langle P_+ f, g_1\rangle$$

Thus $P_+\mathcal{V}_1 f = P_+ f$ for all $f\in L^2(\mathbb{C}_+, d\mu)$ and therefore $P_+\mathcal{V}_1 = P_+$. Now notice that

$$\begin{aligned}(\mathcal{V}_1^2 f)(w) &= \mathcal{V}_1(\mathcal{V}_1 f)(w)\\ &= 3\langle\overline{b}_{\overline{w}}, \hbar_{\overline{\mathcal{V}_1 f}} b_{\overline{w}}\rangle\\ &= 3\langle\overline{b}_{\overline{w}}, \hbar_{\overline{P_+\mathcal{V}_1 f}} b_{\overline{w}}\rangle\\ &= 3\langle\overline{b}_{\overline{w}}, \hbar_{\overline{P_+ f}} b_{\overline{w}}\rangle\\ &= 3\langle\overline{b}_{\overline{w}}, \hbar_{\overline{f}} b_{\overline{w}}\rangle = (\mathcal{V}_1 f)(w)\end{aligned}$$

for all $w\in\mathbb{C}_+$ and $f\in L^2(\mathbb{C}_+, d\mu)$. Hence $\mathcal{V}_1^2 = \mathcal{V}_1$.

Let $\phi\in L^\infty(\mathbb{C}_+)$. The little Hankel operator $\mathcal{S}_{\overline{\phi}}$ defined on $L_a^2(\mathbb{C}_+)$ belong to the class $S_p, 2\le p<\infty$.

Theorem 8.2. *Suppose $f\in L^2(\mathbb{C}_+, d\mu)$. Then $\hbar_{\overline{f}}$ is bounded if and only if $(\mathcal{V}_1 f)(w)$ is bounded in $\mathbb{C}_+$ and there is a constant $C>0$ such that $C^{-1}||\mathcal{V}_1 f||_\infty \le ||\hbar_{\overline{f}}|| \le C||\mathcal{V}_1 f||_\infty$.*

Proof. Notice that $b_{\overline{w}}\in L^2(\mathbb{C}_+, d\mu)$ and $||b_{\overline{w}}||_2 = 1$. Hence $|(\mathcal{V}_1 f)(w)| = 3|\langle\overline{b}_{\overline{w}}, \hbar_{\overline{f}} b_{\overline{w}}\rangle| \le 3||b_{\overline{w}}||_2\ ||\hbar_{\overline{f}}||\ ||b_{\overline{w}}||_2 = 3||b_{\overline{w}}||_2^2\ ||\hbar_{\overline{f}}|| = 3||\hbar_{\overline{f}}||$. Further, $\hbar_{\overline{f}} = \hbar_{\overline{P_+ f}} = \hbar_{\overline{P_+\mathcal{V}_1 f}} = \hbar_{\overline{\mathcal{V}_1 f}}$. Thus $\mathcal{V}_1 f\in L^\infty(\mathbb{C}_+)$ implies that $\hbar_{\overline{f}}$ is bounded with $||\hbar_{\overline{f}}|| \le ||\mathcal{V}_1 f||_\infty$. The result follows since $\hbar_{\overline{f}} = \hbar_{\overline{\mathcal{V}_1 f}}$ for all $f\in L^2(\mathbb{C}_+, d\mu)$. $\square$

Theorem 8.3. *Suppose* $2 \leq p < \infty$. *Then* $\mathcal{S}_{\overline{\phi}} \in S_p$ *if and only if* $\mathcal{V}_1\phi \in L^p(\mathbb{C}_+, d\nu)$, *where* $d\nu(w) = |B(\overline{w}, w)|d\mu(w)$.

Proof. Suppose $2 \leq p < \infty$ and $\mathcal{S}_{\overline{\phi}} \in S_p$. Then

$$\begin{aligned}\int_{\mathbb{C}_+} |(\mathcal{V}_1\phi)(w)|^p d\nu(w) &\leq 3^p \int_{\mathbb{C}_+} ||\mathcal{S}_{\overline{\phi}} b_{\overline{w}}||^p d\nu(w) \\ &= 3^p \int_{\mathbb{C}_+} \langle \mathcal{S}_{\overline{\phi}} b_{\overline{w}}, \mathcal{S}_{\overline{\phi}} b_{\overline{w}}\rangle^{\frac{p}{2}} d\nu(w) \\ &= 3^p \int_{\mathbb{C}_+} \langle \mathcal{S}^*_{\overline{\phi}}\mathcal{S}_{\overline{\phi}} b_{\overline{w}}, b_{\overline{w}}\rangle^{\frac{p}{2}} d\nu(w) \\ &\leq 3^p \int_{\mathbb{C}_+} \langle (\mathcal{S}^*_{\overline{\phi}}\mathcal{S}_{\overline{\phi}})^{\frac{p}{2}} b_{\overline{w}}, b_{\overline{w}}\rangle d\nu(w) \\ &= 3^p \int_{\mathbb{C}_+} \langle |\mathcal{S}_{\overline{\phi}}|^p b_{\overline{w}}, b_{\overline{w}}\rangle d\nu(w) < \infty.\end{aligned}$$

Hence $\mathcal{V}_1\phi \in L^p(\mathbb{C}_+, d\nu)$. Conversely, suppose $\mathcal{V}_1\phi \in L^p(\mathbb{C}_+, d\nu)$. We shall show that $\mathcal{S}_{\overline{\phi}} \in S_p$. Since $\mathcal{S}_{\overline{\phi}} = \mathcal{S}_{\overline{\mathcal{V}_1\phi}}$, it suffices to show that $\mathcal{S}_{\overline{\phi}}$ is in S_p whenever $\phi \in L^p(\mathbb{C}_+, d\nu)$. In the following we prove that if $\phi \in L^p(\mathbb{C}_+, d\nu)$ then $\mathcal{S}_{\overline{\phi}} \in S_p, 1 \leq p < \infty$. From Heinz inequality [10], it follows that

$$|\langle \mathcal{S}_{\overline{\phi}} b_{\overline{w}}, b_{\overline{w_1}}\rangle|^2 \leq \langle |\mathcal{S}_{\overline{\phi}}| b_{\overline{w}}, b_{\overline{w}}\rangle \langle |\mathcal{S}^*_{\overline{\phi}}| b_{\overline{w_1}}, b_{\overline{w_1}}\rangle$$

$$\begin{aligned}&= \langle (\mathcal{S}^*_{\overline{\phi}}\mathcal{S}_{\overline{\phi}})^{\frac{1}{2}} b_{\overline{w}}, b_{\overline{w}}\rangle \langle (\mathcal{S}_{\overline{\phi}}\mathcal{S}^*_{\overline{\phi}})^{\frac{1}{2}} b_{\overline{w_1}}, b_{\overline{w_1}}\rangle \\ &\leq \langle (\mathcal{S}^*_{\overline{\phi}}\mathcal{S}_{\overline{\phi}}) b_{\overline{w}}, b_{\overline{w}}\rangle^{\frac{1}{2}} \langle (\mathcal{S}_{\overline{\phi}}\mathcal{S}^*_{\overline{\phi}}) b_{\overline{w_1}}, b_{\overline{w_1}}\rangle^{\frac{1}{2}} \\ &= ||\mathcal{S}_{\overline{\phi}} b_{\overline{w}}||_2 \; ||\mathcal{S}_{\overline{\phi}^+} b_{\overline{w_1}}||_2 \\ &= ||P_+ J(\overline{\phi} b_{\overline{w}})||_2 \; ||P_+ J(\overline{\phi}^+ b_{\overline{w_1}})||_2 \\ &\leq ||\overline{\phi} b_{\overline{w}}||_2 \; ||\overline{\phi}^+ b_{\overline{w_1}}||_2 \\ &= \left(\int_{\mathbb{C}_+} |\overline{\phi}(u)|^2 |b_{\overline{w}}(u)|^2 d\mu(u)\right)^{\frac{1}{2}} \left(\int_{\mathbb{C}_+} |\overline{\phi}^+(v)|^2 |b_{\overline{w_1}}(v)|^2 d\mu(v)\right)^{\frac{1}{2}} \\ &\leq d\langle \mathcal{T}_{|\phi|} b_{\overline{w}}, b_{\overline{w}}\rangle \langle \mathcal{T}_{|\phi^+|} b_{\overline{w_1}}, b_{\overline{w_1}}\rangle\end{aligned}$$

for some constant $d > 0$. Thus

$$|\langle \mathcal{S}_{\overline{\phi}} B_{\overline{w}}, B_{\overline{w_1}}\rangle|^2 \leq d\langle \mathcal{T}_{|\phi|} B_{\overline{w}}, B_{\overline{w}}\rangle \langle \mathcal{T}_{|\phi^+|} B_{\overline{w_1}}, B_{\overline{w_1}}\rangle.$$

Now $\phi \in L^p(\mathbb{C}_+, d\nu)$ implies $|\phi|, |\phi^+| \in L^p(\mathbb{C}_+, d\nu)$. Hence $\mathcal{T}_{|\phi|}, \mathcal{T}_{|\phi^+|} \in S_p$. Hence by Theorem 7.1, $\mathcal{S}_{\overline{\phi}} \in S_p$. □

Remark 8.1. It follows from Theorem 8.2, that if $f \in L^2(\mathbb{C}_+, d\mu)$ then $\hbar_{\overline{f}} = \hbar_{\overline{P_+ f}} = \hbar_{\overline{P_+ \mathcal{V}_1 f}} = \hbar_{\overline{\mathcal{V}_1 f}}$. Thus $\hbar_{\overline{f}}$ is bounded if and only if $\hbar_f = \hbar_g$ for some $g \in L^\infty(\mathbb{C}_+)$. Suppose $f \in L^2(\mathbb{C}_+, d\mu)$ and $\hbar_{\overline{f}}$ is compact. Then $\mathcal{V}_1 f(w) = 3\langle \overline{b}_{\overline{w}}, \hbar_{\overline{f}} b_{\overline{w}} \rangle \longrightarrow 0$ since $b_{\overline{w}} \longrightarrow 0$ weakly in in $L^2_a(\mathbb{C}_+)$ as $|a| \longrightarrow 1^-$ where $a = M\overline{w}$. From Theorem 8.3, it follows that $\mathcal{V}_1 f \in L^p(\mathbb{C}_+, d\nu)$, if and only if $\hbar_{\overline{f}}$ is in S_p. Since $\hbar_{\overline{f}} = \hbar_{\overline{\mathcal{V}_1 f}}$, it follows that $\hbar_\phi$ is in $S_p, 2 \le p < \infty$ if and only if $\phi \in L^p(\mathbb{C}_+, d\nu)$. This can also be verified as follows: Notice that for $g \in L^2_a(\mathbb{C}_+)$,

$$\hbar_\phi g(w) = \int_{\mathbb{C}_+} \phi(z) g(z) B_{\overline{w}}(z) d\mu(z).$$

Hence

$$\begin{aligned} ||\hbar_\phi||^2_{S_2} &\le \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} |\phi(z)|^2 |B_{\overline{w}}(z)|^2 d\mu(w) d\mu(z) \\ &= \int_{\mathbb{C}_+} |\phi(z)|^2 d\nu(z). \end{aligned}$$

We have seen that $||\hbar_\phi|| \le ||\phi||_\infty$. Thus interpolation gives $||\hbar_\phi||_{S_p} \le ||\phi||_{L^p(\mathbb{C}_+, d\nu)}$ for $2 \le p < \infty$. Thus if $2 \le p < \infty$, then $\hbar_f \in S_p$ if and only if $\hbar_f = \hbar_g$ for some $g \in L^p(\mathbb{C}_+, d\nu)$.

Acknowledgement

The author thanks the referees for their valuable suggestions to improve the paper.

References

[1] **R. Bhatia and P. Semrl**, Distance between hermitian operators in Schatten classes, *Proc. Edinb. Math. Soc.*, **39**, (1996), 377–380.

[2] **M. S. Birman and M. Z. Solomyak**, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel Dordrecht, The Netherlands, 1986.

[3] **S. Elliott and A. Wynn**, Composition operators on weighted Bergman spaces of a half plane, *Proc. Edinb. Math. Soc.*, **54**, (2011), 373–379.

[4] **E. M Ibbaoui and H. Naqos E. O. Fallah**, Composition operators with univalent symbol in Schatten classes, *J. Funct. Anal.*, **3**, (2014), 1547–1564.

[5] **A. Feintuch**, On Hankel operators associated with a class of Non-Toeplitz operators, *J. Funct. Anal.*, **94**, (2009), 1-13.

[6] **T. Furuta**, A simplified proof of Heinz inequality and scrutiny of its equality, *Proc. Amer. Math. Soc.*, **97**, (1986), 751–753.

[7] **T. Furuta**, Reverse inequalities involving two relative operator entropies and two relative entropies, *Linear Algebra Appl.*, **403**, (2005), 24–30.

[8] **A. Karapetyants and N. Vasilevski S. Grudsky**, Dynamics properties of Toeplitz operators on the upper half-plane: parabolic case, *J. Operator Theory*, **52**, (2004), 1–31.

[9] **I. C. Gohberg and M. G. Krein**, *Introduction to the Theory of Linear Non-self adjoint Operators in Hilbert Space*, American Mathematical Society, USA, 1969.

[10] **E. Heinz**, On an inequality for linear operators in a Hilbert space, *Report on Operator Theory and Group Representations, National Academy of Science, USA*, **387**, (1995), 27-29.

[11] **H. S. Kang**, Berezin transforms and Toeplitz operators on the weighted Bergman space of the half plane, *Bull. Korean Math. Soc.*, **40**, (2007), 281–290.

[12] **C. S. Lin**, Polar decomposition approach to Reid's inequality, *J. Inequal. Appl.*, **7**, (2002), 779–785.

[13] **J. Miao and D. Zheng**, Compact operators on Bergman spaces, *Integral Equations Operator Theory*, **48**, (2004), 61-79.

[14] **N. K. Nikolskii**, *Treatise on the shift operator*, Springer -Verlag, Berlin, 1985.

[15] **W. Rudin**, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976.

[16] **V. V. Peller**, *Hankel Operators and Their Applications*, Spinger, Germany, 2003.

[17] **B. Simon**, *Trace Ideals and Their Applications*, American Mathematical Society, USA, 2005.

[18] **D. Walsh and F. Holland**, Hankel operators in Von-Neumann-Schatten classes, *Illinois J. Math.*, **32**, (1988), 1-22.

[19] **K.Zhu**, *Operator Theory in Fuction Spaces*, Marcel Dekker, New York, 1990.

Namita Das

P. G. Dept. of Mathematics, Utkal University, Vani Vihar,
Bhubaneswar- 751004, Odisha, India

E-mail: namitadas440@yahoo.co.in

Jitendra Kumar Behera

P. G. Dept. of Mathematics, Utkal University, Vani Vihar,
Bhubaneswar- 751004, Odisha, India

E-mail: jitendramath0507@gmail.com

Received: 15.02.2017

Accepted: 18.05.2017