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# Hyper Relative Order $(p, q)$ of Entire Functions

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**Abstract.** After the works of Lahiri and Banerjee [6] on the idea of relative order $(p, q)$ of entire functions, we introduce in this paper hyper relative order $(p, q)$ of entire functions where $p$, $q$ are positive integers with $p > q$ and prove sum theorem, product theorem and theorem on derivative.

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## 1 Introduction and Definitions

Let $f$ and $g$ be non-constant entire functions and $M_f(r) = max\{|f(z)| : |z| = r\}$, $M_g(r) = max\{|g(z)| : |z| = r\}$. Then $M_f(r)$ is strictly increasing and continuous function of $r$ and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and $\lim\limits_{R\to\infty} M_f^{-1}(R) = \infty$.

In 1988, Bernal [2] introduced the definition of relative order of $f$ with respect to $g$ as

$$\rho_g(f) = inf\{\mu > 0 : M_f(r) < M_g(r^\mu)$$

for all $r > r_0(\mu) > 0\}$.

When $g(z) = \exp(z)$, $\rho_g(f)$ coincides with the classical definition of order ([15],p-248).

Following Sato [14], we write $\log^{[0]} x = x, \exp^{[0]} x = x$ and for positive integer $m \geq 1, \log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

If $p, q$ are positive integers, $p \geq q$ then Juneja et.al., [7] defined $(p, q)$th order of $f$ by

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{log^{[p]} M_f(r)}{log^{[q]} r}.$$

During the past decades, several authors (see for example [1], [7], [8], [9], [10], [11]) made close investigations on $(p, q)$ order of entire functions. After this in 2005, Lahiri and Banerjee [6] introduced the concept of relative order $(p, q)$ of entire functions as follows.

**Definition 1.1.** [6] *Let $p$ and $q$ be positive integers with $p > q$. The relative order $(p, q)$ of $f$ with respect to $g$ is defined by*

$$\rho_g^{(p,q)}(f) = inf\{\mu > 0 : M_f(r) < M_g(exp^{[p-1]}(\mu log^{[q]} r)) \ for\ all\ r > r_0(\mu) > 0\}$$

$$= \limsup_{r\to\infty} \frac{log^{[p-1]} M_g^{-1} M_f(r)}{log^{[q]} r} = \limsup_{r\to\infty} \frac{log^{[p-1]} M_g^{-1}(r)}{log^{[q]} M_f^{-1}(r)}.$$

If $g(z) = \exp(z)$ then $\rho_g^{(p,q)}(f) = \rho^{(p,q)}(f)$.

In the present paper we introduce the concept of hyper relative $(p, q)$ order as follows.

**Definition 1.2.** *Let $f$ and $g$ be entire functions and $p, q$ are positive integers with $p > q$. The hyper relative $(p, q)$ order of $f$ with respect to $g$ is defined by*

$$\bar{\rho}_g^{(p,q)}(f) = inf\{\mu > 0 : M_f(r) < M_g(exp^{[p]}(\mu log^{[q]} r)) \ for\ all\ r > r_0(\mu) > 0\}.$$

Clearly $\bar{\rho}_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{log^{[p]} M_g^{-1} M_f(r)}{log^{[q]} r}$.

When $p = 2, q = 1$ and $g(z) = \exp(z)$, then the definition coincides with the classical definition of hyper order of entire functions which have been investigated closely by several authors (see for example [4], [13] etc.).

The following definition of Bernal [2] will be needed.

**Definition 1.3.** [2] *A non-constant entire function $g$ is said to have the property $(A)$ if for any $\sigma > 1$ and for all large $r$, $[M_g(r)]^2 \leq M_g(r^\sigma)$ holds.*

Examples of functions with or without the property $(A)$ are avilable in [2]. Throughout the paper we shall assume $f, g, h$ etc., are non-constant entire functions and $M_f(r), M_g(r), M_h(r)$ etc., denote respectively their maximum modulus on $|z| = r$.

## 2 Lemmas

The following lemmas will be needed in the sequel.

**Lemma 2.1.** [2] *Let $g$ be an entire function which satisfies the property $(A)$, and let $\sigma > 1$. Then for any positive integer $n$ and for all large $r$,*

$$[M_g(r)]^n \leq M_g(r^\sigma)$$

*holds.*

**Lemma 2.2.** [2] *Suppose $f$ is an entire function, $\alpha > 1, 0 < \beta < \alpha, s > 1, 0 < \mu < \lambda$ and $n$ is a positive integer. Then*

*(a)* $M_f(\alpha r) > \beta M_f(r)$.

*(b) There exists $K = K(s, f) > 0$ such that $[M_f(r)]^s \leq K M_f(r^s)$ for $r > 0$.*

*(c)* $\lim\limits_{r\to\infty} \frac{M_f(r^s)}{M_f(r)} = \infty = \lim\limits_{r\to\infty} \frac{M_f(r^\lambda)}{M_f(r^\mu)}$.

*(d) If $f$ is transcendental, then*

$$\lim_{r\to\infty} \frac{M_f(r^s)}{r^n M_f(r)} = \infty = \lim_{r\to\infty} \frac{M_f(r^\lambda)}{r^n M_f(r^\mu)}.$$

**Lemma 2.3.** *([12], p-21) Let $f(z)$ be holomorphic in the circle $|z| = 2eR(R > 0)$ with $f(0) = 1$ and $\eta$ be an arbitrary positive number not exceeding $\frac{3e}{2}$. Then inside the circle $|z| = R$, but outside of a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have*

$$\log|f(z)| > -T(\eta) log M_f(2eR)$$

*for $T(\eta) = 2 + \log \frac{3e}{2\eta}$.*

**Lemma 2.4.** [5] *Every entire function $g$ satisfying the property $(A)$ is transcendental.*

**Lemma 2.5.** [3] *Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let $\alpha$ satisfy $0 < \alpha < 1$ and let $C(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$*

$M_{f\circ g}(r) \geq M_f(\, C(\alpha) M_g(\alpha r))$.

*Further if $g(z)$ is any entire function, then with $\alpha = \frac{1}{2}$, for sufficiently large values of $r$*

$M_{f\circ g}(r) \geq M_f(\, \frac{1}{8} M_g(\frac{r}{2}) - |g(0)|)$.

*Clearly*

$$M_{f\circ g}(r) \geq M_f(\frac{1}{16}M_g(\frac{r}{2})). \tag{1}$$

*On the other hand the opposite inequality*

$$M_{f\circ g}(r) \leq M_f(M_g(r)) \tag{2}$$

*is an immediate consequence of the definition.*

**Lemma 2.6.** *If $f$ is a polynomial of degree $n$ and $g$ is transcendental, then $\bar{\rho}_g^{(p,q)}(f) = 0$.*

*Proof.* For all large $r$, $M_f(r) \leq Nr^n$ where $N\ (>0)$ is a constant and $M_g(r) > Kr^m$ where $K\ (>0)$ is a constant and $m(>0)$ is arbitrary.

Then

$$M_g^{-1}M_f(r) < \left(\frac{Nr^n}{K}\right)^{\frac{1}{m}} = \lambda r^{\frac{n}{m}},$$

where $\lambda = \left(\frac{N}{K}\right)^{\frac{1}{m}}$.

Now

$$\bar{\rho}_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1}M_f(r)}{\log^{[q]} r}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p]}\left(\lambda r^{\frac{n}{m}}\right)}{\log^{[q]} r} = 0.$$

□

## 3 Sum Theorem

**Theorem 3.1.** *If $f_1, f_2, g$ and $h$ are entire functions with $0 < \lambda_h \leq \rho_h < \infty$, then for $p > 2$*

$$\bar{\rho}_g^{(p,q)}(f_1 \pm f_2) \leq max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\},$$

*the equality holding when $\rho_{g\circ h}^{(p,q)}(f_1) \neq \rho_{g\circ h}^{(p,q)}(f_2)$.*

*Proof.* We consider the theorem for $f_1 + f_2$. Let $f = f_1 + f_2$ and suppose that

$$\rho_{g\circ h}^{(p,q)}(f_1) \leq \rho_{g\circ h}^{(p,q)}(f_2).$$

Let $\varepsilon > 0$ be arbitrary. For all large r, we have

$$M_{f_1}(r) < M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1) + \varepsilon)\log^{[q]} r\}\right]$$

$$\leq M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

and $M_{f_2}(r) < M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right].$

Now,

$$M_f(r) \leq M_{f_1}(r) + M_{f_2}(r)$$

$$< 2M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

$$< M_{g\circ h}\left[3\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

using Lemma 2.2(a) it follows by (2)

$$M_g\left[M_h\{3\exp^{[p-1]}((\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r)\}\right].$$

So,

$$M_g^{-1}M_f(r) < M_h\left[3\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

i.e.

$$\log M_g^{-1}M_f(r) < \log M_h\left[3\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

$$< \left[3\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]^{(\rho_h + \varepsilon)}$$

i.e.

$$\log\log M_g^{-1}M_f(r) < (\rho_h + \varepsilon)\left[\log 3 + \exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r\}\right]$$

i.e.

$$\log^{[p]} M_g^{-1}M_f(r) < (\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon)\log^{[q]} r + O(1)$$

i.e.

$$\bar{\rho}_g^{(p,q)}(f) \leq (\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon).$$

Since $\varepsilon > 0$ be arbitrary,

$$\overline{\rho}_g^{(p,q)}(f_1+f_2) \le \rho_{g\circ h}^{(p,q)}(f_2) = max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\}. \tag{3}$$

Next let, $\rho_{g\circ h}^{(p,q)}(f_1) < \rho_{g\circ h}^{(p,q)}(f_2)$. Then for all large $r$

$$M_{f_1}(r) < M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1)+\varepsilon)\log^{[q]} r\}\right] \tag{4}$$

and there exists a sequence $\{r_n\}$, $r_n \to \infty$ such that

$$M_{f_2}(r_n) > M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}\right] \text{ for } n=1,2,3,\ldots. \tag{5}$$

From Lemma 2.2(c), we obtain

$$\lim_{r\to\infty} \frac{M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r\}\right]}{M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1)+\varepsilon)\log^{[q]} r\}\right]} = \infty.$$

Then for all large $r$,

$$M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r\}\right]$$

$$> 2M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1)+\varepsilon)\log^{[q]} r\}\right].$$

So,

$$M_{f_2}(r_n) > M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}\right]$$

$$> 2\ M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1)+\varepsilon)\log^{[q]} r_n\}\right].$$

Now for a sequence of values of $\{r_n\}, r_n \to \infty$ we get by using (4)

$$M_{f_2}(r_n) > 2M_{f_1}(r_n) \text{ for } n=1,2,3,\ldots.$$

So,

$$M_f(r_n) \ge M_{f_2}(r_n) - M_{f_1}(r_n) > M_{f_2}(r_n) - \frac{1}{2}M_{f_2}(r_n) = \frac{1}{2}M_{f_2}(r_n)$$

$$> \frac{1}{2}M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}\right],$$

using (5)

$$> M_{goh}\left[\frac{1}{3}\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}\right]$$

using Lemma 2.2(a)

$$\geq M_g\left[\frac{1}{16}M_h\left(\frac{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}}{6}\right)\right],$$

using (1)

i.e.,

$$M_g^{-1}M_f(r_n) > \frac{1}{16}M_h\left[\frac{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}}{6}\right]$$

i.e.,

$$\log M_g^{-1}M_f(r_n) \geq \log M_h\left[\frac{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}}{6}\right] + O(1)$$

$$> \left[\frac{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}}{6}\right]^{(\lambda_h-\varepsilon)} + O(1)$$

i.e.

$$\log\log M_g^{-1}M_f(r_n) > (\lambda_h-\varepsilon)\left[\exp^{[p-2]}\{(\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n\}\right] + O(1).$$

So,

$$\log^{[p]} M_g^{-1}M_f(r_n) > (\rho_{goh}^{(p,q)}(f_2)-\varepsilon)\log^{[q]} r_n + O(1)$$

i.e.,

$$\overline{\rho}_g^{(p,q)}(f) \geq (\rho_{goh}^{(p,q)}(f_2)-\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\rho}_g^{(p,q)}(f_1+f_2) \geq \rho_{goh}^{(p,q)}(f_2) = max\{\rho_{goh}^{(p,q)}(f_1), \rho_{goh}^{(p,q)}(f_2)\}. \tag{6}$$

From (3) and (6) we have

$$\overline{\rho}_g^{(p,q)}(f_1+f_2) = max\{\rho_{goh}^{(p,q)}(f_1), \rho_{goh}^{(p,q)}(f_2)\}.$$

This proves the theorem. □

## 4 Product Theorems

**Theorem 4.1.** *Let $P$ be a polynomial and $f, g, h$ are entire functions with $0 < \lambda_h \leq \rho_h < \infty$, where $f$ is transcendental. Then for $p > 2$*

$$\bar{\rho}_g^{(p,q)}(Pf) = \rho_{goh}^{(p,q)}(f).$$

*Proof.* Let the degree of P(z) be $m$. Then there exists $\alpha$, $0 < \alpha < 1$ and a positive integer $n$ $(> m)$ such that $2\alpha < |P(z)| < r^n$ holds on $|z| = r$, for all large $r$. Now by Lemma 2.2(a)

$$M_f\left(\frac{1}{\alpha}\alpha r\right) > \frac{1}{2\alpha}M_f(\alpha r)$$

i.e.,$M_f(\alpha r) < 2\alpha M_f(r)$.
Let $k(z) = P(z)f(z)$. Then for all large $r$ and $s > 1$
$M_f(\alpha r) < 2\alpha M_f(r) \leq M_k(r) \leq r^n M_f(r) < M_f(r^s)$, by Lemma 2.2(d).
Let $\varepsilon > 0$ be arbitrary. Now for all large $r$,

$$M_k(r) < M_f(r^s)$$

$$< M_{goh}\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}\right]$$

$$< M_g\left[M_h\{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}\}\right] \text{ by (2)}.$$

So, $M_g^{-1}M_k(r) < M_h\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}\right]$
i.e., $\log M_g^{-1}M_k(r) < \log M_h\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}\right]$

$$< \left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}\right]^{(\rho_h+\varepsilon)}$$

i.e., $\log\log M_g^{-1}M_k(r) < (\rho_h + \varepsilon)\exp^{[p-2]}\{(\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s\}$
i.e., $\log^{[p]} M_g^{-1}M_k(r) < (\rho_{goh}^{(p,q)}(f) + \varepsilon)\log^{[q]} r^s + O(1)$.
So, $\bar{\rho}_g^{(p,q)}(k) \leq (\rho_{goh}^{(p,q)}(f) + \varepsilon)$.
Since $\varepsilon > 0$ be arbitrary,

$$\bar{\rho}_g^{(p,q)}(k) \leq \rho_{goh}^{(p,q)}(f). \tag{7}$$

On the other hand for a sequence of values of $\{r_n\}, r_n \to \infty$

$$M_k(r_n) > M_f(\alpha r_n) > M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}\right]$$

$$> M_g\left[\frac{1}{16}M_h\{\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}}{2}\}\right] \text{ by (1).}$$

So, $M_g^{-1}M_k(r_n) > \frac{1}{16}M_h\left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}}{2}\right]$
i.e,

$$\log\ M_g^{-1}M_k(r_n) > \log M_h\left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}}{2}\right] + O(1)$$

$$> \left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}}{2}\right]^{(\lambda_h-\varepsilon)} + O(1)$$

i.e, $\log\log\ M_g^{-1}M_k(r_n) > (\lambda_h-\varepsilon)\left[\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n)\}\right] + O(1)$

i.e, $\log^{[p]}\ M_g^{-1}M_k(r_n) > (\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}(\alpha r_n) + O(1)$
i.e., $\bar{\rho}_g^{(p,q)}(k) \geq (\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)$.
Since $\varepsilon > 0$ is arbitrary,

$$\bar{\rho}_g^{(p,q)}(k) \geq \rho_{g\circ h}^{(p,q)}(f). \tag{8}$$

From (7) and (8) we have

$$\bar{\rho}_g^{(p,q)}(k) = \rho_{g\circ h}^{(p,q)}(f)$$

i.e.,$\bar{\rho}_g^{(p,q)}(Pf) = \rho_{g\circ h}^{(p,q)}(f)$. □

**Theorem 4.2.** *If $n > 1$ be a positive integer and $f, g$ and $h$ are entire functions with $0 < \lambda_h \leq \rho_h < \infty$, then for $p > 2$*

$$\bar{\rho}_g^{(p,q)}(f^n) = \rho_{g\circ h}^{(p,q)}(f).$$

*Proof.* From Lemmas 2.2(a) and 2.2(b), we obtain

$$[M_f(r)]^n \le K M_f(r^n)$$

$$< M_f[(K+1)r^n] < M_{goh}\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\right]$$

$$< M_g\left[M_h\{\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\}\right], \text{ by}(2)$$

where $K = K(n,f) > 0, n > 1, r > 1$.

So,

$$M_g^{-1}[M_f(r)]^n < M_h\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\right]$$

i.e.,

$$\log M_g^{-1}[M_f(r)]^n < \log M_h\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\right]$$

$$< \left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\right]^{(\rho_h+\varepsilon)}$$

i.e., $\log\log M_g^{-1}[M_f(r)]^n < (\rho_h+\varepsilon)\left[\exp^{[p-2]}\{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)\}\right]$

i.e., $\log^{[p]} M_g^{-1}[M_f(r)]^n < (\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n) + O(1)$

i.e, $\limsup\limits_{r\to\infty}\dfrac{\log^{[p]} M_g^{-1}[M_f(r)]^n}{\log^{[q]} r}$

$\le \limsup\limits_{r\to\infty}\frac{(\rho_{goh}^{(p,q)}(f)+\varepsilon)\log^{[q]}((K+1)r^n)}{\log^{[q]}((K+1)r^n)}\limsup\limits_{r\to\infty}\frac{\log^{[q]}((K+1)r^n)}{\log^{[q]} r^n}$

So, $\bar{\rho}_g^{(p,q)}(f^n) \le (\rho_{goh}^{(p,q)}(f)+\varepsilon)$.

Since $\varepsilon > 0$ be arbitrary

$$\bar{\rho}_g^{(p,q)}(f^n) \le \rho_{goh}^{(p,q)}(f). \tag{9}$$

On the other hand for a sequence of values of $r = r_n$

$$[M_f(r_n)]^n > M_f(r_n)$$

$$> M_{goh}\left[\exp^{[p-1]}\{(\rho_{goh}^{(p,q)}(f)-\varepsilon)\log^{[q]}(r_n)\}\right]$$

$$> M_g \left[ \frac{1}{16} M_h \{ \frac{\exp^{[p-1]}\{(\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n)\}}{2} \} \right], \text{ by}(1)$$

i.e., $M_g^{-1} \left[ M_f(r_n) \right]^n > \frac{1}{16} M_h \left[ \frac{\exp^{[p-1]}\{(\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n)\}}{2} \right]$

i.e.,

$$\log M_g^{-1} \left[ M_f(r_n) \right]^n > \log M_h \left[ \frac{\exp^{[p-1]}\{(\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n)\}}{2} \right] + O(1)$$

$$> \left[ \frac{\exp^{[p-1]}\{(\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n)\}}{2} \right]^{(\lambda_h - \varepsilon)} + O(1)$$

i.e., $\log \log M_g^{-1} \left[ M_f(r_n) \right]^n > (\lambda_h - \varepsilon) \left[ \exp^{[p-2]}\{(\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n)\} \right] + O(1)$

i.e., $\log^{[p]} M_g^{-1} \left[ M_f(r_n) \right]^n > (\rho_{g \circ h}^{(p,q)}(f) - \varepsilon) \log^{[q]}(r_n) + O(1)$.

So, $\overline{\rho}_g^{(p,q)}(f^n) \geq (\rho_{g \circ h}^{(p,q)}(f) - \varepsilon)$.

Since $\varepsilon > 0$ be arbitrary,

$$\overline{\rho}_g^{(p,q)}(f^n) \geq \rho_{g \circ h}^{(p,q)}(f). \tag{10}$$

From (9) and (10) we have,

$$\overline{\rho}_g^{(p,q)}(f^n) = \rho_{g \circ h}^{(p,q)}(f).$$

□

**Theorem 4.3.** *If $f_1, f_2, g$ and $h$ are entire functions with $0 < \lambda_h \leq \rho_h < \infty$, where $g$ is transcendental and $g \circ h$ has the property (A) then for $p > 2$*

*(i)* $\overline{\rho}_g^{(p,q)}(f_1 f_2) \leq \max\{\rho_{g \circ h}^{(p,q)}(f_1), \rho_{g \circ h}^{(p,q)}(f_2)\}$.

*(ii) Equality holds if* $\rho_{g \circ h}^{(p,q)}(f_1) \neq \rho_{g \circ h}^{(p,q)}(f_2)$.

*Proof.* By Lemma 2.4, $g \circ h$ is transcendental. We consider the following three cases.

Case(a). $f_1$ and $f_2$ both are polynomials. Then by Lemma 2.6

$$\overline{\rho}_g^{(p,q)}(f_1 f_2) \leq \max\{\rho_{g \circ h}^{(p,q)}(f_1), \rho_{g \circ h}^{(p,q)}(f_2)\}.$$

Case(b). $f_1$ is polynomial and $f_2$ is transcendental. Then by Theorem 4.1

$$\bar{\rho}_g^{(p,q)}(f_1 f_2) \le \max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\}.$$

Case(c). $f_1$ and $f_2$ both are transcendental. Let $\rho_{g\circ h}^{(p,q)}(f_1) \le \rho_{g\circ h}^{(p,q)}(f_2)$ and $k = f_1 f_2$. Let $\varepsilon > 0$ be arbitrary. For all large $r$, we have

$$M_{f_1}(r) < M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_1) + \frac{\varepsilon}{2}) \log^{[q]} r\}\right]$$

$$\le M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r\}\right]$$

and $M_{f_2}(r) < M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r\}\right]$.

Now,

$$\frac{\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} r\}}{\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r\}} \to \infty$$

as r$\to \infty$.

So for all large $r$, say $r \ge r_1 > r_0$ the above expression is greater than

$$\frac{\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} r_0\}}{\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r_0\}} = \sigma$$

(say).

Then $\sigma > 1$. For all large r, we have,

$$M_k(r) \le M_{f_1}(r) M_{f_2}(r) < \left(M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r\}\right]\right)^2$$

$$< \left(M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} r\}\right]^{\sigma}\right)$$

since $g \circ h$ has the property (A) and $\sigma > 1$

$$< M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} r\}\right]$$

for $r \ge r_1 > r_0$

$$\le M_g\left[M_h\{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} r\}\}\right] \text{ by (2).}$$

So,

$$M_g^{-1} M_k(r) \le M_h\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} r\}\right]$$

i.e.,

$$\log M_g^{-1} M_k(r) \leq \log M_h \left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)+\varepsilon)\log^{[q]} r\}\right]$$

$$< \left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)+\varepsilon)\log^{[q]} r\}\right]^{(\rho_h+\varepsilon)}$$

i.e.,

$$\log\log M_g^{-1} M_k(r) < (\rho_h+\varepsilon)\log \left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f_2)+\varepsilon)\log^{[q]} r\}\right]$$

i.e.,

$$\log^{[p]} M_g^{-1} M_k(r) < (\rho_{g\circ h}^{(p,q)}(f_2)+\varepsilon)\log^{[q]} r + O(1).$$

So, $\limsup\limits_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_k(r)}{\log^{[q]} r} \leq \rho_{g\circ h}^{(p,q)}(f_2)+\varepsilon$.

Since $\varepsilon > 0$ be arbitrary,

$$\bar{\rho}_g^{(p,q)}(k) \leq \rho_{g\circ h}^{(p,q)}(f_2) = \max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\}. \tag{11}$$

(ii) Suppose $\rho_{g\circ h}^{(p,q)}(f_1) < \rho_{g\circ h}^{(p,q)}(f_2)$. Without loss of generality we may assume $f_1(0) = 1$.

We choose $\mu, \lambda$ so that $\rho_{g\circ h}^{(p,q)}(f_1) < \mu < \lambda < \rho_{g\circ h}^{(p,q)}(f_2)$. There exists a sequence $R_n \to \infty$ such that

$$M_{f_2}(R_n) > M_{g\circ h}[exp^{[p-1]}(\lambda log^{[q]} R_n)], \text{ for } n = 1, 2, 3 \ldots.$$

Also for all large $r$,

$$M_{f_1}(r) < M_{g\circ h}[exp^{[p-1]}(\mu log^{[q]} r)].$$

In Lemma 2.3, we take $f_1(z)$ for $f(z), \eta = \frac{1}{16}, R = 2R_n$ and obtain

$$log|f_1(z)| > -T(\eta) log M_{f_1}(2e.2R_n),$$

where

$$T(\eta) = 2 + log(\frac{3e}{2.\frac{1}{16}}) = 2 + log(24e).$$

So,

$$log|f_1(z)| > -(2 + log(24e)) log M_{f_1}(4eR_n)$$

holds within and on $|z| = 2R_n$ but outside a family of excluded circles the sum of whose radii is not greater than $4.\frac{1}{16}.2R_n$ i.e., $\frac{R_n}{2}$.

If $r_n \in (R_n, 2R_n)$ then on $|z| = r_n$, $log|f_1(z)| > -7 log M_{f_1}(4eR_n)$.

Now,

$$M_{f_2}(r_n) > M_{f_2}(R_n) > M_{g\circ h}\left[\exp^{[p-1]}(\lambda \log^{[q]} R_n)\right]$$

$$> M_{g\circ h}\left[\exp^{[p-1]}(\lambda \log^{[q]} \frac{r_n}{2})\right].$$

Let $z_r$ be a point on $|z| = r_n$ such that $M_{f_2}(r_n) = |f_2(z_r)|$.
Then

$$M_k(r_n) = max\{|k(z)| : |z| = r_n\}$$

$$= max\{|f_1(z)||f_2(z)| : |z| = r_n\}$$

$$> [M_{f_1}(4eR_n)]^{-7} M_{g\circ h}[exp^{[p-1]}(\lambda log^{[q]} \frac{r_n}{2})]$$

$$> \left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4eR_n))\}\right]^{-7} M_{g\circ h}[exp^{[p-1]}(\lambda log^{[q]} \frac{r_n}{2})]$$

$$\geq \left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4er_n))\}\right]^{-7} M_{g\circ h}[exp^{[p-1]}(\lambda log^{[q]} \frac{r_n}{2})],$$

since $r_n > R_n$.
Now for all large $n$, we have $\frac{log^{[q]} 4er_n}{log^{[q]} \frac{r_n}{2}} < \frac{\lambda}{\gamma}$ where $\gamma \in (\mu, \lambda)$. So,

$$M_k(r_n) > \left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4er_n))\}\right]^{-7} M_{g\circ h}[exp^{[p-1]}(\gamma log^{[q]} 4er_n)]. \tag{12}$$

The expression

$$\frac{exp^{[p-2]}(\gamma log^{[q]}(4er_n))}{exp^{[p-2]}(\mu log^{[q]}(4er_n))}$$

tends to infinity as $n \to \infty$.

So for all large $n$, $r_n \geq r_1 > r_0$ we may write

$$\frac{exp^{[p-2]}(\gamma log^{[q]}(4er_n))}{exp^{[p-2]}(\mu log^{[q]}(4er_n))} > \frac{exp^{[p-2]}(\gamma log^{[q]}(4er_0))}{exp^{[p-2]}(\mu log^{[q]}(4er_0))} = \alpha$$

(say), then $\alpha > 1$.

From Lemma 2.1 and for all large r, we have

$$\left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4er_n))\}^{\alpha}\right]$$

$$> \left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4er_n))\}\right]^8. \tag{13}$$

Also for the above value of $\alpha$, one can easily verify that

$$\left[M_{g\circ h}\{exp^{[p-1]}(\gamma log^{[q]}(4er_n))\}\right]$$

$$> \left[M_{g\circ h}\{exp^{[p-1]}(\mu log^{[q]}(4er_n))\}^{\alpha}\right]. \tag{14}$$

Therefore for all large n, we have from (12), (13) and (14)

$$M_k(r_n) > M_{g\circ h}\left[exp^{[p-1]}(\mu log^{[q]}(4er_n))\right]$$

$$> M_{g\circ h}\left[exp^{[p-1]}(\mu log^{[q]} r_n)\right]$$

$$\geq M_g\left[\frac{1}{16}M_h\{\frac{exp^{[p-1]}(\mu log^{[q]} r_n)}{2}\}\right] \text{ by (1).}$$

So,

$$M_g^{-1}M_k(r_n) > \frac{1}{16}M_h\left[\frac{exp^{[p-1]}(\mu log^{[q]} r_n)}{2}\right]$$

i.e.,

$$\log M_g^{-1}M_k(r_n) > \log M_h\left[\frac{exp^{[p-1]}(\mu log^{[q]} r_n)}{2}\right] + O(1)$$

$$> \left[\frac{exp^{[p-1]}(\mu log^{[q]} r_n)}{2}\right]^{(\lambda_h-\varepsilon)} + O(1)$$

i.e.,

$$\log\log M_g^{-1}M_k(r_n) > (\lambda_h-\varepsilon)\left[exp^{[p-2]}(\mu log^{[q]} r_n)\right] + O(1)$$

i.e., $\log^{[p]} M_g^{-1}M_k(r_n) > \mu\log^{[q]} r_n + O(1)$.

So, $\overline{\rho}_g^{(p,q)}(f_1 f_2) \geq \mu$.

Now since $\mu < \rho_{g\circ h}^{(p,q)}(f_2)$ is arbitrary, we have

$$\overline{\rho}_g^{(p,q)}(f_1 f_2) \geq \rho_{g\circ h}^{(p,q)}(f_2) = \max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\}. \tag{15}$$

From (11) and (15), we have

$$\overline{\rho}_g^{(p,q)}(f_1 f_2) = \max\{\rho_{g\circ h}^{(p,q)}(f_1), \rho_{g\circ h}^{(p,q)}(f_2)\}.$$

This proves the theorem. □

## 5 Hyper Relative order (p,q) of the derivative

**Theorem 5.1.** *Let $f$, $g$ and $h$ be entire transcendental with $0 < \lambda_h \leq \rho_h < \infty$. Then for $p > 2$*

$$\bar{\rho}_g^{(p,q)}(f') = \rho_{g \circ h}^{(p,q)}(f).$$

*Proof.* We write $M_{f'}(r) = max\{|f'(z)| : |z| = r\}, M_{g'}(r) = max\{|g'(z)| : |z| = r\}$ and $M_{h'}(r) = max\{|h'(z)| : |z| = r\}$.

Without loss of generality we may assume that $f(0) = 0$. Otherwise we set $f_1(z) = zf(z)$. Then $f_1(0) = 0$ and by Theorem 4.1
$\bar{\rho}_g^{(p,q)}(f_1) = \rho_{g \circ h}^{(p,q)}(f)$.
We may write $f(z) = \int_0^z f'(t)dt$, where the line of integration is the segment from $z = 0$ to $z = re^{i\theta_0}, r > 0$. Let $z_1 = re^{i\theta_1}$ be such that $|f(z_1)| = max\{|f(z)| : |z| = r\}$. Then

$$M_f(r) = |f(z_1)| = |\int_0^{z_1} f'(t)dt| \leq rM_{f'}(r). \tag{16}$$

Let $C$ denote the circle $|t - z_0| = r$, where $z_0, |z_0| = r$ is defined so that

$$|f'(z_0)| = max\{|f'(z)| : |z| = r\}.$$

So,

$$M_{f'}(r) = max\{|f'(z)| : |z| = r\} =$$

$$|f'(z_0)| = |\frac{1}{2\pi i} \oint_C \frac{f(t)}{(t - z_0)^2} dt| \leq \frac{1}{2\pi} \frac{M_f(2r)}{r^2} 2\pi r = \frac{M_f(2r)}{r}. \tag{17}$$

From (16) and (17) we obtain

$$\frac{M_f(r)}{r} \leq M_{f'}(r) \leq \frac{M_f(2r)}{r}, \text{ for } r > 0. \tag{18}$$

Let $\sigma \in (0, 1)$. From Lemma 2.2(d), $\lim_{r\to\infty} \frac{M_f(r^s)}{r^n M_f(r)} = \infty$, where $s > 1$ and $n$ is a positive integer and so $M_f(r^s) > r^n M_f(r)$ for all large $r$. If we replace $r$ by $r^\sigma$ and $s = \frac{1}{\sigma}$ then from above $M_f(r^{s\sigma}) > r^{n\sigma} M_f(r^\sigma) \geq rM_f(r^\sigma)$,where the positive integer $n$ is such that $n\sigma \geq 1$.

So $M_f(r) > rM_f(r^\sigma)$ for all large $r$.
From (18) and above, we have

$$M_f(r^\sigma) < \frac{M_f(r)}{r} \leq M_{f'}(r) \leq \frac{M_f(2r)}{r} < M_f(2r) \text{ for all large } r > 1. \quad (19)$$

So, $M_{f'}(r) < M_f(2r)$

$$< M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\right]$$

$$\leq M_g\left[M_h\{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\}\right], \text{ by (2)}$$

i.e.,

$$M_g^{-1}M_{f'}(r) < M_h\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\right]$$

i.e.,

$$\log M_g^{-1}M_{f'}(r) < \log M_h\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\right]$$

$$< \left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\right]^{(\rho_h+\varepsilon)}$$

i.e.,

$$\log\log M_g^{-1}M_{f'}(r) < (\rho_h+\varepsilon)\left[\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r\}\right]$$

i.e.,

$$\log^{[p]} M_g^{-1}M_{f'}(r) < (\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r + O(1).$$

So,

$$\limsup_{r\to\infty}\frac{\log^{[p]} M_g^{-1}M_{f'}(r)}{\log^{[q]} r} \leq \lim_{r\to\infty}\frac{(\rho_{g\circ h}^{(p,q)}(f)+\varepsilon)\log^{[q]} 2r}{\log^{[q]} 2r}.\lim_{r\to\infty}\frac{\log^{[q]} 2r}{\log^{[q]} r}$$

i.e.,

$$\bar{\rho}_g^{(p,q)}(f') \leq \rho_{g\circ h}^{(p,q)}(f)+\varepsilon.$$

Since $\varepsilon > 0$ be arbitrary,

$$\bar{\rho}_g^{(p,q)}(f') \leq \rho_{g\circ h}^{(p,q)}(f). \quad (20)$$

Now for a sequence of values of $r = r_n$, we have from (19)

$$M_{f'}(r_n) > M_f(r_n^\sigma) > M_{g\circ h}\left[\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]} r_n^\sigma\}\right]$$

$$> M_g\left[\frac{1}{16}M_h\{\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma}\}}{2}\}\right], \text{by } (1)$$

i.e.,

$$M_g^{-1}M_{f'}(r_n) > \frac{1}{16}M_h\left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma}\}}{2}\right]$$

i.e.,

$$\log M_g^{-1}M_{f'}(r_n) > \log M_h\left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma}\}}{2}\right] + O(1) >$$

$$\left[\frac{\exp^{[p-1]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma}\}}{2}\right]^{(\lambda_h-\varepsilon)} + O(1)$$

i.e.,

$$\log\log M_g^{-1}M_{f'}(r_n) > (\lambda_h-\varepsilon)\left[\exp^{[p-2]}\{(\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma}\}\right] + O(1)$$

i.e.,

$$\log^{[p]}M_g^{-1}M_{f'}(r_n) > (\rho_{g\circ h}^{(p,q)}(f)-\varepsilon)\log^{[q]}r_n^{\sigma} + O(1).$$

So, $\overline{\rho}_g^{(p,q)}(f') \geq \rho_{g\circ h}^{(p,q)}(f)-\varepsilon$.
Since $\varepsilon > 0$ be arbitrary,

$$\overline{\rho}_g^{(p,q)}(f') \geq \rho_{g\circ h}^{(p,q)}(f). \tag{21}$$

From (20) and (21), we have

$$\overline{\rho}_g^{(p,q)}(f') = \rho_{g\circ h}^{(p,q)}(f).$$

This proves the theorem. □

## 6 Conclusion

Our main goal through this paper is to enquire the basic relation between hyper relative (p,q) orders of entire function with respect to a single entire function and also of composition of entire functions which have not studied previously. But still there remains some problems to be investigated for future researchers in this field.

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