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On Almost Generalized Weakly Symmetric LP-Sasakian Manifolds

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Abstract. The purpose of this paper is to introduce the notions of an almost generalized weakly symmetric LP-Sasakian manifolds and an almost generalized weakly Ricci-symmetric LP-Sasakian manifolds. The existence of an almost generalized weakly symmetric LP-Sasakian manifold is ensured by a non-trivial example.

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1 Introduction

The notion of a weakly symmetric Riemannian manifold was initiated by Tamássy and Binh [16]. In analogy to [12], a weakly symmetric Riemannian manifold $(M^n, g)(n > 2)$, is said to be an almost weakly pseudo symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and satisfies the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y, U, V, W) = {} & [A_1(X) + B_1(X)]\bar{R}(Y, U, V, W) \\ & + C_1(Y)\bar{R}(X, U, V, W) + C_1(U)\bar{R}(Y, X, V, W) \\ & + D_1(V)\bar{R}(Y, U, X, W) + D_1(W)\bar{R}(Y, U, V, X)\end{aligned} \tag{1.1}$$

where A_1, B_1, C_1 & D_1 are non-zero 1-forms defined by $A_1(X) = g(X, \sigma_1)$, $B_1(X) = g(X, \varrho_1)$, $C_1(X) = g(X, \pi_1)$ and $D_1(X) = g(X, \partial_1)$, for all X and $\bar{R}(Y, U, V, W) = g(R(Y, U)V, W)$, ∇ being the operator of the covariant differentiation with respect to the metric tensor g. An n-dimensional Riemannian manifold of this kind is denoted by $A(WPS)_n$-manifold.

Keeping the tune of Dubey [14], we shall call a Riemannian manifold of dimension n, an almost generalized weakly symmetric (which is abbreviated hereafter as $A(GWS)_n$- manifold) if it admits the equation

$$\begin{aligned} & (\nabla_X \bar{R})(Y, U, V, W) \\ = \; & [A_1(X) + B_1(X)]\bar{R}(Y, U, V, W) + C_1(Y)\bar{R}(X, U, V, W) \\ & + C_1(U)\bar{R}(Y, X, V, W) + D_1(V)\bar{R}(Y, U, X, W) \\ & + D_1(W)\bar{R}(Y, U, V, X) + [A_2(X) + B_2(X)]\bar{G}(Y, U, V, W) \\ & + C_2(Y)\bar{G}(X, U, V, W) + C_2(U)\,\bar{G}(Y, X, V, W) \\ & + D_2(V)\,\bar{G}(Y, U, X, W) + D_2(W)\,\bar{G}(Y, U, V, X) \end{aligned} \tag{1.2}$$

where

$$\bar{G}(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)] \tag{1.3}$$

and A_i, B_i, C_i & D_i are non-zero 1-forms defined by $A_i(X) = g(X, \sigma_i)$, $B_i(X) = g(X, \varrho_i)$, $C_i(X) = g(X, \pi_i)$ and $D_i(X) = g(X, \partial_i)$, for $i = 1, 2$. The beauty of such $A(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space in the sense of Cartan (for $A_i = B_i = C_i = D_i = 0$),

(ii) recurrent space by Walker [13](for $A_1 \neq 0$, $A_2 = B_i = C_i = D_i = 0$),

(iii) generalized recurrent space by Dubey [14] ($A_i \neq 0$ and $B_i = C_i = D_i = 0$),

(iv) pseudo symmetric space by Chaki [11] (for $A_1 = B_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$),

(v) semi-pseudo symmetric space in the sense of Tarafder et al. [10] (for $A_1 = -B_1, C_1 = D_1$ and $A_2 = B_2 = C_2 = D_2 = 0$),

(vi) generalized semi-pseudo symmetric space in the sense of Baishya [6] (for $A_1 = -B_1, C_1 = D_1$ and $A_2 = -B_2, C_2 = D_2$),

(vii) generalized pseudo symmetric space, by Baishya [5] (for $A_i = B_i = C_i = D_i \neq 0$),

(viii) almost pseudo symmetric space in the sprite of Chaki et al. [12] (for $B_1 \neq 0$, $A_1 = C_1 = D_1 \neq 0$ and $A_2 = B_2 = C_2 = D_2 = 0$),

(ix) almost generalized pseudo symmetric space in the sense of Baishya (for $B_i \neq 0$, $A_i = C_i = D_i \neq 0$) and

(x) weakly symmetric space by Tamássy and Binh [16] (for $A_1 = A_2 = B_2 = C_2 = D_2 = 0$).

In this connection we would like to mention our work in [7], [8] and [9].

Our work is structured as follows. Section 2 is concerned with LP-Sasakian manifolds and some known results. In section 3, we have investigated an almost generalized weakly symmetric LP-Sasakian manifold and obtained some interesting results. Section 4, is concerned with an almost generalized weakly Ricci-symmetric LP-Sasakian manifold. Finally, we have constructed an example of an almost generalized weakly symmetric LP-Sasakian manifold.

2 LP-Sasakian manifolds and some known results

In 1989 K. Matsumoto ([1]) introduced the notion of Lorentzian para-Sasakian (LP-Sasakian for short) manifold. In 1992, Mihai and Rosca([15]) defined the same notion independently. This type of manifold is also discussed in ([2], [3])

An n-dimensional differentiable manifold M is said to be an LP-Sasakian manifold [1] if it admits a $(1, 1)$ tensor field ϕ, a unit timelike contravarit vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \qquad \nabla_X \xi = \phi X, \tag{2.2}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold :

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad Rank\ \phi = n - 1. \tag{2.4}$$

Again, if we put

$$\Omega(X, Y) = g(X, \phi Y)$$

for any vector fields X, Y then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field ([15]). Also, since the vector field η is closed in an LP-Sasakian manifold, we have ([1], [15])

$$(\nabla_X \eta)(Y) = \Omega(X, Y), \qquad \Omega(X, \xi) = 0 \tag{2.5}$$

for any vector fields X and Y.

Let M be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then the following relations hold ([1], [15]) :

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.6}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$S(X,\xi) = (n-2)\eta(X), \tag{2.9}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-2)\eta(X)\eta(Y), \tag{2.10}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor of the manifold.

3 Almost generalized weakly symmetric LP - Sasakian manifold

An LP-Sasakian manifold $(M^n, g)(n > 2)$, is said to be an almost generalized weakly symmetric if it admits the relation (1.2).

Now, contracting Y over W in both sides of (1.2), we get

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = \; & [A_1(X) + B_1(X)]S(U,V) + C_1(U)S(X,V) \\ & + C_1(R(X,U)V) + D_1(R(X,V)U) + D_1(V)S(U,X) \\ & + (n-1)[\{A_2(X) + B_2(X)\}g(U,V) + C_2(U)g(X,V)] \\ & + D_2(V)(n-1)g(U,X) + C_2(G(X,U)V) + D_2(G(X,V)U) \end{aligned} \tag{3.1}$$

which yields

$$\begin{aligned} & (\nabla_X S)(U,\xi) \\ = \; & [A_1(X) + B_1(X)](n-2)\eta(U) + C_1(U)(n-2)\eta(X) \\ & + D_1(\xi)S(U,X) + \eta(U)C_1(X) - \eta(X)C_1(U) + \eta(U)D_1(X) \\ & - g(X,U)D_1(\xi) + (n-1)[\{A_2(X) + B_2(X)\}\eta(U) \\ & + C_2(U)\eta(X) + D_2(\xi)g(U,X)] + \eta(U)C_2(X) \\ & - \eta(X)C_2(U) + \eta(U)D_2(X) - g(U,X)D_2(\xi) \end{aligned} \tag{3.2}$$

for $V = \xi$. Again, replacing V by ξ, in the following identity

$$(\nabla_X S)(U,V) = \nabla_X S(U,V) - S(\nabla_X U,V) - S(U,\nabla_X V) \tag{3.3}$$

and then making use of (2.2), (2.5), (2.9), we find

$$(\nabla_X S)(U,\xi) = (n-2)g(X,\phi U) - S(U,\phi X). \tag{3.4}$$

Next, in consequence of (3.2) and (3.4), we have

$$\begin{aligned} & (n-2)g(X,\phi U) - S(U,\phi X) \\ = \; & (n-2)[\{A_1(X) + B_1(X)\}\eta(U) + C_1(U)\eta(X)] + D_1(\xi)S(U,X) \\ & + \eta(U)C_1(X) - \eta(X)C_1(U) + \eta(U)D_1(X) - g(X,U)D_1(\xi) \\ & + (n-1)[\{A_2(X) + B_2(X)\}\eta(U) + C_2(U)\eta(X) + D_2(\xi)g(U,X)] \\ & + \eta(U)C_2(X) - \eta(X)C_2(U) + \eta(U)D_2(X) - g(U,X)D_2(\xi). \end{aligned} \tag{3.5}$$

Next, putting $X = U = \xi$ in (3.5) and using (2.1), (2.2) and (2.9), we get

$$\begin{aligned} & (n-2)[A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi)] \\ + \quad & (n-1)[A_2(\xi) + B_2(\xi) + C_2(\xi) + D_2(\xi)] = 0. \end{aligned} \tag{3.6}$$

In particular, if $A_2(\xi) = B_2(\xi) = C_2(\xi) = D_2(\xi) = 0$, (3.6) becomes

$$A_1(\xi) + B_1(\xi) + C_1(\xi) + D_1(\xi) = 0. \tag{3.7}$$

This leads to the followings

Theorem 3.1. *In an almost generalized weakly symmetric LP-Sasakian manifold $(M^n, g)(n > 2)$, the relation (3.6) hold good.*

Corollary 3.2. *In an almost weakly symmetric LP-Sasakian manifold $(M^n, g)(n > 2)$, the relation (3.7) hold good.*

By virtue of (2.1), (2.2) and (2.9), the equation (3.1) turns into

$$\begin{aligned} & (\nabla_X S)(\xi, V) \\ = \; & (n-2)[\{A_1(X) + B_1(X)\}\eta(V) + D_1(V)\eta(X)] + C_1(\xi)S(X,V) \\ & + \eta(V)C_1(X) - g(X,V)C_1(\xi) + \eta(V)D_1(X) - \eta(X)D_1(V) \\ & + (n-1)[\{A_2(X) + B_2(X)\}\eta(V) + D_2(V)\eta(X)] \\ & + C_2(\xi)(n-1)g(X,V) + \eta(V)C_2(X) - g(X,V)C_2(\xi) \\ & + \eta(V)D_2(X) - \eta(X)D_2(V) \end{aligned} \tag{3.8}$$

for $U = \xi$. Again, putting $U = \xi$ in (3.3) and using (2.2), (2.5), (2.9), we obtain

$$(\nabla_X S)(\xi, V) = (n-2)g(X,\phi V) - S(V,\phi X). \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\begin{aligned}
& (n-2)g(X,\phi V)-S(V,\phi X) \\
= \; & (n-2)[\{A_1(X)+B_1(X)\}\eta(V)+D_1(V)\eta(X)]+C_1(\xi)S(X,V) \\
& +\eta(V)C_1(X)-g(X,V)C_1(\xi)+\eta(V)D_1(X)-\eta(X)D_1(V) \\
& +(n-1)[\{A_2(X)+B_2(X)\}\eta(V)+D_2(V)\eta(X)] \\
& +C_2(\xi)(n-1)g(X,V)+\eta(V)C_2(X)-g(X,V)C_2(\xi) \\
& +\eta(V)D_2(X)-\eta(X)D_2(V). \qquad (3.10)
\end{aligned}$$

Putting $V=\xi$ in (3.10) and using (2.1), (2.2), (2.9), we obtain

$$\begin{aligned}
0 \; = \; & -(n-2)\{A_1(X)+B_1(X)\}-C_1(X)-D_1(X) \\
& +(n-3)D_1(\xi)\eta(X)+(n-3)C_1(\xi)\eta(X) \\
& -(n-1)\{A_2(X)+B_2(X)\}-C_2(X)-D_2(X) \\
& +(n-2)D_2(\xi)\eta(X)+(n-2)C_2(\xi)\eta(X). \qquad (3.11)
\end{aligned}$$

Putting $X=\xi$ in (3.10) and using (2.1), (2.2), (2.9), we obtain

$$\begin{aligned}
0 \; = \; & (n-2)\{A_1(\xi)+B_1(\xi)\}\eta(V)+(n-2)C_1(\xi)\eta(V) \\
& +\eta(V)D_1(\xi)-(n-3)D_1(V)+ \\
& (n-1)\{A_2(\xi)+B_2(\xi)\}\eta(V)+(n-1)C_2(\xi)\eta(V) \\
& +\eta(V)D_2(\xi)-(n-2)D_2(V). \qquad (3.12)
\end{aligned}$$

Replacing V by X in the above equation

$$\begin{aligned}
0 \; = \; & (n-2)\{A_1(\xi)+B_1(\xi)\}\eta(X)-(n-3)D_1(X) \\
& +C_1(\xi)(n-2)\eta(X)+\eta(X)D_1(\xi) \\
& +(n-1)\{A_2(\xi)+B_2(\xi)\}\eta(X)-(n-2)D_2(X) \\
& +(n-1)C_2(\xi)\eta(X)+\eta(X)D_2(\xi). \qquad (3.13)
\end{aligned}$$

Adding (3.11) and (3.13), we get

$$\begin{aligned}
0 \; = \; & -(n-2)\{A_1(X)+B_1(X)+D_1(X)\}-C_1(X) \\
& -(n-1)\{A_2(X)+B_2(X)+D_2(X)\}-C_2(X) \\
& +(n-2)C_2(\xi)\eta(X)+(n-3)C_1(\xi)\eta(X) \qquad (3.14)
\end{aligned}$$

Putting $X=\xi$ in (3.5)

$$\begin{aligned}
0 \; = \; & [A_1(\xi)+B_1(\xi)](n-2)\eta(U)-C_1(U)(n-2)+D_1(\xi)(n-2)\eta(U) \\
& +\eta(U)C_1(\xi)+C_1(U)+\eta(U)D_1(\xi)-\eta(U)D_1(\xi) \\
& +(n-1)[\{A_2(\xi)+B_2(\xi)\}\eta(U)-C_2(U)+D_2(\xi)\eta(U)] \\
& +\eta(U)C_2(\xi)+C_2(U)+\eta(U)D_2(\xi)-\eta(U)D_2(\xi). \qquad (3.15)
\end{aligned}$$

Replacing U by X in (3.15), we get

$$\begin{aligned} & [(n-2)\{A_1(\xi)+B_1(\xi)+D_1(\xi)\}+C_1(\xi)]\eta(X)-(n-3)C_1(X) \\ = & (n-2)C_2(X)-[(n-1)\{A_2(\xi)+B_2(\xi)+D_2(\xi)\}+C_2(\xi)]\eta(X). \end{aligned} \tag{3.16}$$

In view of (3.6), above equation becomes

$$\begin{aligned} & (n-3)C_1(X)+(n-2)C_2(X) \\ = & -(n-3)C_1(\xi)\eta(X)-(n-2)C_2(\xi)\eta(X). \end{aligned} \tag{3.17}$$

Subtracting (3.17) from (3.14), we have,

$$\begin{aligned} 0 = & -(n-2)\{A_1(X)+B_1(X)+C_1(X)+D_1(X)\} \\ & -(n-1)\{A_2(X)+B_2(X)+C_2(X)+D_2(X)\}. \end{aligned} \tag{3.18}$$

Theorem 3.3. *In an almost generalized weakly symmetric LP-Sasakian manifold* $(M^n, g)(n > 2)$, *the sum of the associated 1-forms is given by (3.18).*

Next, in view of $A_2 = B_2 = C_2 = D_2 = 0$, the relation (3.18) yields

$$A_1(X)+B_1(X)+C_1(X)+D_1(X)=0. \tag{3.19}$$

This motivates us to state

Theorem 3.4. *In a weakly symmetric LP-Sasakian manifold* $(M^n, g)(n > 2)$, *the sum of the associated 1-forms is given by (3.19).*

Theorem 3.5. *There does not exist an LP-Sasakian manifold which is*

(i) recurrent,

(ii) generalized recurrent provided the vector fields associated to the 1-forms are colinear,

(iii) pseudo symmetric,

(iv) generalized semi-pseudo symmetric provided the vector fields associated to the 1-forms are colinear,

(v) generalized almost pseudo symmetric provided the the vector fields associated to the 1-forms are colinear.

4 Almost generalized weakly Ricci-symmetric LP-Sasakian manifold

An LP-Sasakian manifold $(M^n, g)(n > 2)$, is said to be almost generalized weakly Ricci-symmetric if there exist 1-forms $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$ which satisfy the condition

$$\begin{aligned}
& (\nabla_X S)(U,V) \\
= \; & [\bar{A}_1(X)+\bar{B}_1(X)]S(U,V)+\bar{C}_1(U)S(X,V)+\bar{D}_1(V)S(U,X) \\
& +[\bar{A}_2(X)+\bar{B}_2(X)]g(U,V)+\bar{C}_2(U)g(X,V)+\bar{D}_2(V)g(U,X). \quad (4.1)
\end{aligned}$$

Putting $V=\xi$ in (4.1), we obtain

$$\begin{aligned}
& (\nabla_X S)(U,\xi) \\
= \; & [\bar{A}_1(X)+\bar{B}_1(X)](n-2)\eta(U)+\bar{C}_1(U)(n-2)\eta(X)+\bar{D}_1(\xi)S(U,X) \\
& +[\bar{A}_2(X)+\bar{B}_2(X)]\eta(U)+\bar{C}_2(U)\eta(X)+\bar{D}_2(\xi)g(U,X). \quad (4.2)
\end{aligned}$$

In view of (3.4), the relation (4.2) becomes

$$\begin{aligned}
& (n-2)g(X,\phi U)-S(U,\phi X) \\
= \; & [\bar{A}_1(X)+\bar{B}_1(X)](n-2)\eta(U)+\bar{C}_1(U)(n-2)\eta(X)+\bar{D}_1(\xi)S(U,X) \\
& +[\bar{A}_2(X)+\bar{B}_2(X)]\eta(U)+\bar{C}_2(U)\eta(X)+\bar{D}_2(\xi)g(U,X). \quad (4.3)
\end{aligned}$$

Setting $X=U=\xi$ in (4.3) and using (2.1), (2.2) and (2.9), we get

$$\begin{aligned}
& (n-2)[\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{C}_1(\xi)+\bar{D}_1(\xi)] \\
+ \; & [\bar{A}_2(\xi)+\bar{B}_2(\xi)+\bar{C}_2(\xi)+\bar{D}_2(\xi)]=0. \quad (4.4)
\end{aligned}$$

Again, putting $X=\xi$ in (4.3), we get

$$\begin{aligned}
& (n-2)\bar{C}_1(U)+\bar{C}_2(U) \\
= \; & [\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{D}_1(\xi)](n-2)\eta(U)+[\bar{A}_2(\xi)+\bar{B}_2(\xi)+\bar{D}_2(\xi)]\eta(U). \quad (4.5)
\end{aligned}$$

Setting $U=\xi$ in (4.3) and then using (2.1), (2.2) and (2.9), we obtain from (4.3) that

$$\begin{aligned}
& (n-2)[\bar{A}_1(X)+\bar{B}_1(X)]+[\bar{A}_2(X)+\bar{B}_2(X)] \\
= \; & \bar{C}_1(\xi)(n-2)\eta(X)+\bar{D}_1(\xi)(n-2)\eta(X)+\bar{C}_2(\xi)\eta(X)+\bar{D}_2(\xi)\eta(X). \quad (4.6)
\end{aligned}$$

Replacing U by X in (4.5) and adding with (4.6), we have

$$\begin{aligned}
& (n-2)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{C}_1(X)]+[\bar{A}_2(X)+\bar{B}_2(X)+\bar{C}_2(X)] \\
= \; & [\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{D}_1(\xi)+\bar{C}_1(\xi)](n-2)\eta(X)+[\bar{A}_2(\xi)+\bar{B}_2(\xi)+\bar{D}_2(\xi) \\
& +\bar{C}_2(\xi)]\eta(X)+\bar{D}_1(\xi)(n-2)\eta(X)+\bar{D}_2(\xi)\eta(X). \quad (4.7)
\end{aligned}$$

In consequence of (4.4), the above equation becomes

$$\begin{aligned}
& (n-2)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{C}_1(X)]+[\bar{A}_2(X)+\bar{B}_2(X)+\bar{C}_2(X)] \\
= \; & \bar{D}_1(\xi)(n-2)\eta(X)+\bar{D}_2(\xi)\eta(X). \quad (4.8)
\end{aligned}$$

Putting $X = U = \xi$ in (4.1), we get

$$\begin{aligned} & \bar{D}_1(V)(n-2) + \bar{D}_2(V) \\ = \; & [\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{C}_1(\xi)](n-2)\eta(V) + [\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{C}_2(\xi)]\eta(V) \end{aligned} \tag{4.9}$$

Replacing V by X in (4.9) and adding with (4.8), we obtain

$$\begin{aligned} & (n-2)[\bar{A}_1(X) + \bar{B}_1(X) + \bar{C}_1(X) + \bar{D}_1(X)] \\ & +[\bar{A}_2(X) + \bar{B}_2(X) + \bar{C}_2(X) + \bar{D}_1(X)] \\ = \; & (n-2)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{C}_1(\xi) + \bar{D}_1(\xi)]\eta(V) \\ & +[\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{C}_2(\xi) + \bar{D}_2(\xi)]\eta(V). \end{aligned} \tag{4.10}$$

By virtue of (4.4), the above equation becomes

$$\begin{aligned} & (n-2)[\bar{A}_1(X) + \bar{B}_1(X) + \bar{C}_1(X) + \bar{D}_1(X)] \\ = \; & -[\bar{A}_2(X) + \bar{B}_2(X) + \bar{C}_2(X) + \bar{D}_1(X)]. \end{aligned} \tag{4.11}$$

This leads to the followings

Theorem 4.1. *In an almost generalized weakly Ricci symmetric LP-Sasakian manifold $(M^n, g)(n > 2)$, the sum of the associated 1-forms are related by (4.11).*

Theorem 4.2. *There does not exist an LP-Sasakian manifold which is*

(i) Ricci recurrent,

(ii) generalized Ricci recurrent provided the the vector fields associated to the 1-forms are colinear,

(iii) pseudo Ricci-symmetric,

(iv) generalized semi Ricci-pseudo symmetric provided the the vector fields associated to the 1-forms are colinear,

(v) generalized almost Ricci-pseudo symmetric provided the the vector fields associated to the 1-forms are colinear.

5 Example of an A(GWS)$_3$ LP-Sasakian manifold

(see [4], p-286-287) Let $M^3(\phi, \xi, \eta, g)$ be an LP-Sasakian manifold (M^3, g) with a ϕ-basis

$$e_1 = e^z \frac{\partial}{\partial x}, \; e_2 = \phi e_1 = e^{z-\alpha x} \frac{\partial}{\partial y}, \; e_3 = \xi = \frac{\partial}{\partial z},$$

where α is non-zero constant. Then from Koszul's formula for Lorentzian metric g, we can obtain the Levi-Civita connection as follows

$$\begin{aligned} \nabla_{e_1} e_3 &= e_2, & \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 &= e_1, & \nabla_{e_2} e_2 = \alpha e^z e_3, && \nabla_{e_2} e_1 = \alpha e^z e_2, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\begin{aligned} \bar{R}(e_1, e_2, e_1, e_2) &= -(1 - \alpha^2 e^{2z}) \\ \bar{R}(e_1, e_3, e_1, e_3) &= 1 = \bar{R}(e_2, e_3, e_2, e_3). \end{aligned}$$

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X,\ Y,\ U,\ V \in \chi(M)$ can be written as

$$X = \sum_1^3 a_i e_i,\ Y = \sum_1^3 b_i e_i,\ U = \sum_1^3 c_i e_i,\ \ V = \sum_1^3 d_i e_i,$$

$$\begin{aligned} \bar{R}(X, Y, U, V) &= -(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)(1 - \alpha^2 e^{2z}) \\ +(a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2) &= T_1 \text{ (say)} \end{aligned}$$

$$\bar{R}(e_1, Y, U, V) = b_3(c_1 d_3 - c_3 d_1) - b_2\ (c_1 d_2 - c_2 d_1)(1 - \alpha^2 e^{2z}) = \lambda_1 \text{ (say)}$$

$$\bar{R}(e_2, Y, U, V) = b_3(c_2 d_3 - c_3 d_2) + b_1\ (c_1 d_2 - c_2 d_1)(1 - \alpha^2 e^{2z}) = \lambda_2 \text{ (say)}$$

$$\bar{R}(e_3, Y, U, V) = b_1(c_3 d_{1-} c_1 d_3\ ) + b_2\ (c_3 d_2 - c_2 d_3) = \lambda_3 \text{ (say)}$$

$$\bar{R}(X, e_1, U, V) = a_3(c_1 d_3 - c_3 d_1) - a_2\ (c_1 d_2 - c_2 d_1)(1 - \alpha^2 e^{2z}) = \lambda_4 \text{ (say)}$$

$$\bar{R}(X, e_2, U, V) = a_3(c_2 d_3 - c_3 d_2) + a_1\ (c_1 d_2 - c_2 d_1)(1 - \alpha^2 e^{2z}) = \lambda_5 \text{ (say)}$$

$$\bar{R}(X, e_3, U, V) = a_1(c_3 d_{1-} c_1 d_3\ ) + a_2\ (c_3 d_2 - c_2 d_3\ ) = \lambda_6 \text{ (say)}$$

$$\bar{R}(X, Y, e_1, V) = d_3(a_1 b_3 - a_3 b_1) - d_2\ (a_1 b_2 - a_2 b_1\ )(1 - \alpha^2 e^{2z}) = \lambda_7 \text{ (say)}$$

$$\bar{R}(X, Y, e_2, V) = d_3(a_2 b_3 - a_3 b_2) + d_1\ (a_1 b_2 - a_2 b_1\ )(1 - \alpha^2 e^{2z}) = \lambda_8 \text{ (say)}$$

$$\bar{R}(X, Y, e_3, V) = d_1(a_3 b_1 - a_1 b_3\ ) + d_2\ (a_3 b_2 - a_2 b_3) = \lambda_9 \text{ (say)}$$

$$\bar{R}(X, Y, U, e_1) = c_3(a_3 b_1 - a_1 b_3\ ) + c_2\ (a_1 b_2 - a_2 b_1\ )(1 - \alpha^2 e^{2z}) = \lambda_{10} \text{ (say)}$$

$$\bar{R}(X, Y, U, e_2) = c_3(a_3 b_2 - a_2 b_3\ ) - c_1\ (a_1 b_2 - a_2 b_1\ )(1 - \alpha^2 e^{2z}) = \lambda_{11} \text{ (say)}$$

$$\bar{R}(X, Y, U, e_3) = c_1(a_1 b_3 - a_3 b_1) + c_2\ (a_2 b_3 - a_3 b_2) = \lambda_{12} \text{ (say)}$$

$$\begin{aligned}\bar{G}(X,Y,U,V) &= (b_1c_1+b_2c_2-b_3c_3)(a_1d_1+a_2d_2-a_3d_3)\\ &\quad -(a_1c_1+a_2c_2-a_3c_3)(b_1d_1+b_2d_2-b_3d_3)\\ &= T_2 \text{ (say)}\end{aligned}$$

$$\bar{G}(e_1,Y,U,V)=(b_2c_2-b_3c_3)d_1-(b_2d_2-b_3d_3)c_1=\omega_1 \text{ (say)}$$

$$\bar{G}(e_2,Y,U,V)=(b_1c_1-b_3c_3)d_2-(b_1d_1-b_3d_3)c_2=\omega_2 \text{ (say)}$$

$$\bar{G}(e_3,Y,U,V)=-(b_1c_1+b_2c_2)d_3+(b_1d_1+b_2d_2)c_3=\omega_3 \text{ (say)}$$

$$\bar{G}(X,e_1,U,V)=(a_2d_2-a_3d_3)c_1-(a_2c_2-a_3c_3)d_1=\omega_4 \text{ (say)}$$

$$\bar{G}(X,e_2,U,V)=(a_1d_1-a_3d_3)c_2-(a_1c_1-a_3c_3)d_2=\omega_5 \text{ (say)}$$

$$\bar{G}(X,e_3,U,V)=-(a_1d_1+a_2d_2)c_3+(a_1c_1+a_2c_2)d_3=\omega_6 \text{ (say)}$$

$$\bar{G}(X,Y,e_1,V)=(a_2d_2-a_3d_3)b_1-(b_2d_2-b_3d_3)a_1=\omega_7 \text{ (say)}$$

$$\bar{G}(X,Y,e_2,V)=(a_1d_1-a_3d_3)b_2-(b_1d_1-b_3d_3)a_2=\omega_8 \text{ (say)}$$

$$\bar{G}(X,Y,e_3,V)=(b_1d_1+b_2d_2)a_3-(a_1d_1+a_2d_2)b_3=\omega_9 \text{ (say)}$$

$$\bar{G}(X,Y,U,e_1)=(b_2c_2-b_3c_3)a_1-(a_2c_2-a_3c_3)b_1,=\omega_{10} \text{ (say)}$$

$$\bar{G}(X,Y,U,e_2)=(b_1c_1-b_3c_3)a_2-(a_1c_1-a_3c_3)b_2=\omega_{11} \text{ (say)}$$

$$\bar{G}(X,Y,U,e_3)=-(b_1c_1+b_2c_2)a_3+(a_1c_1+a_2c_2)b_3=\omega_{12} \text{ (say)}$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$\begin{aligned}&(\nabla_{e_1}\bar{R})(X,Y,U,V)\\ =\ & a_1\lambda_3-a_3\lambda_2+b_1\lambda_6-b_3\lambda_5+c_1\lambda_9-c_3\lambda_8+d_1\lambda_{12}-d_3\lambda_{11},\end{aligned}$$

$$\begin{aligned}&(\nabla_{e_2}\bar{R})(X,Y,U,V)\\ =\ & -\alpha e^z a_1\lambda_2-\alpha e^z a_2\lambda_3-a_3\lambda_1-\alpha e^z b_1\lambda_5-\alpha e^z b_2\lambda_6-b_3\lambda_4-\alpha e^z c_1\lambda_8\\ & -\alpha e^z c_2\lambda_9-c_3\lambda_7-\alpha e^z d_1\lambda_{11}-\alpha e^z d_2\lambda_{12}-d_3\lambda_{10},\end{aligned}$$

$$(\nabla_{e_3}\bar{R})(X,Y,U,V)=2\alpha^2e^{2z}(b_2-a_2b_1)(c_1d_2-c_2d_1).$$

For the following choice of the the one forms

$$
\begin{aligned}
A_1(e_1) &= \frac{a_1\lambda_3 \ - a_3\lambda_2}{T_1}, B_1(e_1) = \frac{b_1\lambda_6 \ - b_3\lambda_5}{T_1}, \\
A_2(e_1) &= \frac{c_1\lambda_9 \ - c_3\lambda_8}{T_2}, B_2(e_1) = \frac{d_1\lambda_{12} \ - d_3\lambda_{11}}{T_2}, \\
A_1(e_2) &= -\frac{\alpha e^z a_1\lambda_2 + \alpha e^z a_2\lambda_3 + a_3\lambda_1}{T_1}, \\
B_1(e_2) &= -\frac{\alpha e^z b_1\lambda_5 + \alpha e^z b_2\lambda_6 + b_3\lambda_4}{T_1}, \\
A_2(e_2) &= -\frac{\alpha e^z c_1\lambda_8 + \alpha e^z c_2\lambda_9 + c_3\lambda_7}{T_2}, \\
B_2(e_2) &= -\frac{\alpha e^z d_1\lambda_{11} + \alpha e^z d_2\lambda_{12} + d_3\lambda_{10}}{T_2}, \\
A_1(e_3) &= \frac{\alpha^2 e^{2z}(a_1b_2 \ - a_2b_1)c_1d_2}{T_1}, \\
B_1(e_3) &= \frac{-\alpha^2 e^{2z}(a_1b_2 \ - a_2b_1)c_2d_1}{T_1}, \\
C_1(e_3) &= \frac{1}{a_3\lambda_3 + b_3\lambda_6}, \\
C_2(e_3) &= \frac{1}{a_3\theta_3 + b_3\theta_6}, \\
D_1(e_3) &= -\frac{1}{c_3\lambda_9 + d_3\lambda_{12}}, \\
D_2(e_3) &= -\frac{1}{c_3\theta_9 + d_3\theta_{12}}, \\
A_2(e_3) &= \frac{\alpha^2 e^{2z}(a_1b_2 \ - a_2b_1)c_1d_2}{T_2}, \\
B_2(e_3) &= -\frac{\alpha^2 e^{2z}(a_1b_2 \ - a_2b_1)c_2d_1}{T_2},
\end{aligned}
$$

one can easily verify the relations

$$
\begin{aligned}
(\nabla_{e_i}\bar{R})(X,Y,U,V) &= [A_1(e_i) + B_1(e_i)]\bar{R}(X,Y,U,V) \\
&\quad +C_1(X)\bar{R}(e_i,Y,U,V) + C_1(Y)\bar{R}(X,e_i,U,V) \\
&\quad +D_1(U)\bar{R}(X,Y,e_i,V) + D_1(V)\bar{R}(X,Y,U,e_i) \\
&\quad +[A_2(e_i) + B_2(e_i)]\bar{G}(X,Y,U,V) \\
&\quad +C_2(X)\bar{G}(e_i,Y,U,V) + C_2(Y)\bar{G}(X,e_i,U,V) \\
&\quad +D_2(U)\bar{G}(X,Y,e_i,V) + D_2(V)\bar{G}(X,Y,U,e_i)
\end{aligned}
$$

for$1,2,3$. From the above, we can state that

Theorem 5.1. *There exist an LP-Sasakian manifold* (M^3, g) *which is an almost generalized weakly symmetry LP-Sasakian manifold.*

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