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# Iterative Convergence with Banach Space Valued Functions in Abstract Fractional Calculus

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**Abstract.** The goal of this paper is to present a semi-local convergence analysis for some iterative methods under generalized conditions. The operator is only assumed to be continuous and its domain is open. Applications are suggested including Banach space valued functions of fractional calculus, where all integrals are of Bochner-type.

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## 1 Introduction

Let $B_1, B_2$ stand for Banach space and let $\Omega$ stand for an open subset of $B_1$. Let also $U(z, \rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z, \rho)$ stand for the closure of $U(z, \rho)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can be brought in a form like

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[16], where $F : \Omega \to B_2$ is a continuous operator. The solution $x^*$ of equation (1.1) is sought in closed form. However, this is attainable only in special cases, which explains why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving equation (1.1). We can divide these methods in two categories.

**Explicit Methods** [6, 7, 11, 15, 16]: Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot, \cdot; F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [7, 15, 16].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from $B_1$ into $B_2$. Other explicit methods can be found in [7], [11], [15], [16] and the references there in.

**Implicit Methods** [6, 9, 11, 16]:

$$F(x_n) + A_n(x_{n+1} - x_n) = 0 \tag{1.5}$$

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.6}$$

where $A_n = A(x_{n+1}, x_n) = A(F)(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$. We denote $A(F)(x, x) = A(x, x) = A(x)$ for each $x \in \Omega$.

There is a plethora on local as well as semi-local convergence results for explicit methods [1]-[8], [10]-[16]. However, the research on the convergence of implicit methods has received little attention. Authors, usually consider the fixed point problem

$$P_z(x) = x, \tag{1.7}$$

where

$$P_z(x) = x + F(z) + A(x, z)(x - z) \tag{1.8}$$

or

$$P_z(x) = z - A(x, z)^{-1} F(z) \tag{1.9}$$

for methods (1.5) and (1.6), respectivelly, where $z \in \Omega$ is given. If $P$ is a contraction operator mapping a closed set into itself, then according to the contraction mapping principle [11], [12], [15], [16], $P_z$ has a fixed point $x_z^*$

which can be found using the method of succesive substitutions or Picard's method [16] defined for each fixed $n$ by

$$y_{k+1,n} = P_{x_n}\left(y_{k,n}\right), \quad y_{0,n} = x_n, \;\; x_{n+1} = \lim_{k\to+\infty} y_{k,n}. \tag{1.10}$$

Let us also consider the analogous explicit methods

$$F\left(x_n\right) + A\left(x_n, x_n\right)\left(x_{n+1} - x_n\right) = 0 \tag{1.11}$$

$$x_{n+1} = x_n - A\left(x_n, x_n\right)^{-1} F\left(x_n\right) \tag{1.12}$$

$$F\left(x_n\right) + A\left(x_n, x_{n-1}\right)\left(x_{n+1} - x_n\right) = 0 \tag{1.13}$$

and

$$x_{n+1} = x_n - A\left(x_n, x_{n-1}\right)^{-1} F\left(x_n\right). \tag{1.14}$$

In the present paper in Section 2, we present the semi-local convergence of method (1.5) and method (1.6). Section 3 contains the semi-local convergence of method (1.11), method (1.12), method (1.13) and method (1.14). Some applications to Abstract Fractional Calculus are suggested in Section 4 on a certain Banach space valued functions, where all the integrals are of Bochner-type [7], [13].

## 2 Semi-local Convergence for Implicit methods

We present the semi-local convergence analysis of method (1.6) using conditions $(S)$:

$(s_1)$ $F : \Omega \subset B_1 \to B_2$ is continuous and $A(x,y) \in \mathcal{L}(B_1, B_2)$ for each $(x,y) \in \Omega \times \Omega$.

$(s_2)$ There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

$(s_3)$ There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \to [0,+\infty)$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(x,y)(x-y)\| \leq$$

$$\beta\psi\left(\|x-y\|, \|x-x_0\|, \|y-x_0\|\right)\|x-y\|.$$

$(s_4)$ For each $x \in \Omega_0$ there exists $y \in \Omega_0$ such that

$$y = x - A(y, x)^{-1} F(x).$$

$(s_5)$ For $x_0 \in \Omega_0$ and $x_1 \in \Omega_0$ satisfying $(s_4)$ there exists $\eta \geq 0$ such that

$$\left\|A(x_1, x_0)^{-1} F(x_0)\right\| \leq \eta.$$

$(s_6)$ Define $q(t) := \psi(\eta, t, t)$ for each $t \in [0, +\infty)$. Equation

$$t(1 - q(t)) - \eta = 0$$

has positive solutions. Denote by $s$ the smallest such solution.

$(s_7)$ $\overline{U}(x_0, s) \subset \Omega$, where

$$s = \frac{\eta}{1 - q_0} \quad \text{and} \quad q_0 = \psi(\eta, s, s).$$

Next, we present the semi-local convergence analysis for method (1.6) using the conditions $(S)$ and the preceding notation.

**Theorem 2.1.** *Assume that the conditions $(S)$ hold. Then, sequence $\{x_n\}$ generated by method (1.6) starting at $x_0 \in \Omega$ is well defined in $U(x_0, s)$, remains in $U(x_0, s)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \overline{U}(x_0, s)$ of equation $F(x) = 0$. Moreover, suppose that there exists a continuous and nondecreasing function $\psi_1 : [0, +\infty)^4 \to [0, +\infty)$ such that for each $x, y, z \in \Omega_1$*

$$\|F(x) - F(y) - A(z, y)(x - y)\| \leq$$

$$\beta\psi_1(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|)\|x - y\|$$

*and $q_1 = \psi_1(\eta, s, s, s) < 1$.*

*Then, $x^*$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, s)$.*

*Proof.* By the definition of $s$ and $(s_5)$, we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on $k$. Suppose that $\|x_k - x_{k-1}\| \leq q_0^{k-1}\eta$ and $\|x_k - x_0\| \leq s$.

We get by (1.6), $(s_2) - (s_5)$ in turn that

$$\|x_{k+1} - x_k\| = \left\|A_k^{-1} F(x_k)\right\| = \left\|A_k^{-1}(F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1}))\right\|$$

$$\leq \left\|A_k^{-1}\right\| \|F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \leq$$

$$\beta^{-1}\beta\psi(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|y_k - x_0\|)\|x_k - x_{k-1}\| \leq$$

$$\psi\left(\eta,s,s\right)\left\|x_{k}-x_{k-1}\right\|=q_{0}\left\|x_{k}-x_{k-1}\right\|\leq q_{0}^{k}\left\|x_{1}-x_{0}\right\|\leq q_{0}^{k}\eta \tag{2.1}$$

and

$$\begin{aligned}\left\|x_{k+1}-x_{0}\right\| &\leq \left\|x_{k+1}-x_{k}\right\|+...+\left\|x_{1}-x_{0}\right\|\\ &\leq q_{0}^{k}\eta+...+\eta=\frac{1-q_{0}^{k+1}}{1-q_{0}}\eta<\frac{\eta}{1-q_{0}}=s.\end{aligned}$$

The induction is completed. Moreover, we have by (2.1) that for $m=0,1,2,...$

$$\left\|x_{k+m}-x_{k}\right\|\leq\frac{1-q_{0}^{m}}{1-q_{0}}q_{0}^{k}\eta.$$

It follows from the preceding inequation that sequence $\{x_k\}$ is complete in a Banach space $B_1$ and as such it converges to some $x^{*}\in\overline{U}\left(x_{0},s\right)$ (since $\overline{U}\left(x_{0},s\right)$ is a closed ball). By letting $k\to+\infty$ in (2.1) we get $F\left(x^{*}\right)=0$. To show the uniqueness part, let $x^{**}\in U\left(x_{0},s\right)$ be a solution of equation $F\left(x\right)=0$. By using (1.6) and the hypothesis on $\psi_1$, we obtain in turn that

$$\left\|x^{**}-x_{k+1}\right\|=\left\|x^{**}-x_{k}+A_{k}^{-1}F\left(x_{k}\right)-A_{k}^{-1}F\left(x^{**}\right)\right\|\leq$$

$$\left\|A_{k}^{-1}\right\|\left\|F\left(x^{**}\right)-F\left(x_{k}\right)-A_{k}\left(x^{**}-x_{k}\right)\right\|\leq$$

$$\beta^{-1}\beta\psi_{1}\left(\left\|x^{**}-x_{k}\right\|,\left\|x_{k-1}-x_{0}\right\|,\left\|x_{k}-x_{0}\right\|,\left\|x^{**}-x_{0}\right\|\right)\left\|x^{**}-x_{k}\right\|\leq$$

$$q_{1}\left\|x^{**}-x_{k}\right\|\leq q_{1}^{k+1}\left\|x^{**}-x_{0}\right\|,$$

so $\lim_{k\to+\infty}x_{k}=x^{**}$. We have shown that $\lim_{k\to+\infty}x_{k}=x^{*}$, so $x^{*}=x^{**}$. □

**Remark 2.1.** (1) The equation in $(s_6)$ is used to determine the smallness of $\eta$. It can be replaced by a stronger condition as follows. Choose $\mu\in(0,1)$. Denote by $s_0$ the smallest positive solution of equation $q\left(t\right)=\mu$. Notice that if function $q$ is strictly increasing, we can set $s_{0}=q^{-1}\left(\mu\right)$. Then, we can suppose instead of $(s_6)$ :

$(s_6')$ $\eta\leq\left(1-\mu\right)s_{0}$

which is a stronger condition than $(s_6)$.

However, we wanted to leave the equation in $(s_6)$ as uncluttered and as weak as possible.

(2) Condition $(s_2)$ can become part of condition $(s_3)$ by considering

$(s_3)'$ There exists a continuous and nondecreasing function $\varphi:[0,+\infty)^{3}\to[0,+\infty)$ such that for each $x,y\in\Omega_1$

$$\left\|A\left(x,y\right)^{-1}\left[F\left(x\right)-F\left(y\right)-A\left(x,y\right)\left(x,y\right)\right]\right\|\leq$$

$$\varphi\left(\left\|x-y\right\|,\left\|x-x_{0}\right\|,\left\|y-x_{0}\right\|\right)\left\|x-y\right\|.$$

Notice that

$$\varphi(u_1, u_2, u_3) \leq \psi(u_1, u_2, u_3)$$

for each $u_1 \geq 0$, $u_2 \geq 0$ and $u_3 \geq 0$. Similarly, a function $\varphi_1$ can replace $\psi_1$ for the uniqueness of the solution part. These replacements are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in $(s_6)$, error bounds and the precision of $s$.

(3) Suppose that there exist $\beta > 0$, $\beta_1 > 0$ and $L \in \mathcal{L}(B_1, B_2)$ with $L^{-1} \in \mathcal{L}(B_2, B_1)$ such that

$$\left\| L^{-1} \right\| \leq \beta^{-1}$$

$$\| A(x, y) - L \| \leq \beta_1$$

and

$$\beta_2 := \beta^{-1}\beta_1 < 1.$$

Then, it follows from the Banach lemma on invertible operators [11], and

$$\left\| L^{-1} \right\| \| A(x, y) - L \| \leq \beta^{-1}\beta_1 = \beta_2 < 1$$

that $A(x, y)^{-1} \in \mathcal{L}(B_2, B_1)$. Let $\beta = \frac{\beta^{-1}}{1-\beta_2}$. Then, under these replacements, condition $(s_2)$ is implied, therefore it can be dropped from the conditions $(S)$.

(4) Clearly method (1.5) converges under the conditions $(S)$, since (1.6) implies (1.5).

(5) We wanted to leave condition $(s_4)$ as uncluttered as possible, since in practice equations (1.6) (or (1.5)) may be solvable in a way avoiding the already mentioned conditions of the contraction mapping principle. However, in what follows we examine the solvability of method (1.5) under a stronger version of the contraction mapping principle using the conditions $(V)$ :

$(v_1) = (s_1)$.

$(v_2)$ There exist functions $w_1 : [0, +\infty)^4 \to [0, +\infty)$, $w_2 : [0, +\infty)^4 \to [0, +\infty)$ continuous and nondecreasing such that for each $x, y, z \in \Omega$

$$\| I + A(x, z) - A(y, z) \| \leq w_1 (\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|)$$

$$\| A(x, z) - A(y, z) \| \leq w_2 (\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|) \, ||x - y||$$

and

$$w_1(0, 0, 0, 0) = w_2(0, 0, 0, 0) = 0.$$

Set

$$h(t, t, t, t) = \begin{cases} w_1(2t, t, t, t) + w_2(2t, t, t, t)(t + \|x_0\|), & z \neq x_0 \\ w_1(2t, t, t, 0) + w_2(2t, t, t, 0)\|x_0\|, & z = x_0. \end{cases}$$

$(v_3)$ There exists $\tau > 0$ satisfying

$$h(t,t,t,t) < 1$$

and

$$h(t,t,0,t)\,t + \|F(x_0)\| \le t$$

$(v_4)$ $\overline{U}(x_0,\tau) \subseteq D$.

**Theorem 2.2.** *Suppose that the conditions $(V)$ are satisfied. Then, equation (1.5) is uniquely solvable for each $n = 0,1,2,\ldots$. Moreover, if $A_n^{-1} \in \mathcal{L}(B_2,B_1)$, the equation (1.6) is also uniquely solvable for each $n = 0,1,2,\ldots$*

*Proof.* The result is an application of the contraction mapping principle. Let $x,y,z \in U(x_0,\tau)$. By the definition of operator $P_z$, $(v_2)$ and $(v_3)$, we get in turn that

$$\|P_z(x) - P_z(y)\| = \|(I + A(x,z) - A(y,z))(x-y) - (A(x,z) - A(y,z))\,z\|$$

$$\le \|I + A(x,z) - A(y,z)\|\,\|x-y\| + \|A(x,z) - A(y,z)\|\,\|z\|$$

$$\le [w_1(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|) +$$

$$w_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|)(\|z-x_0\| + \|x_0\|)]\,\|x-y\|$$

$$\le h(\tau,\tau,\tau,\tau)\,\|x-y\|$$

and

$$\|P_z(x) - x_0\| \le \|P_z(x) - P_z(x_0)\| + \|P_z(x_0) - x_0\|$$

$$\le h(\|x-x_0\|, \|x-x_0\|, 0, \|z-x_0\|)\,\|x-x_0\| + \|F(x_0)\|$$

$$\le h(\tau,\tau,0,\tau)\,\tau + \|F(x_0)\| \le \tau.$$

□

**Remark 2.2.** Section 2 and Section 3 have an interest independent of Section 4. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract Fractional Calculus as illustrated in Section 4. By specializing function $\psi$, we can apply the results of say Theorem 2.1 in the examples suggested in Section 4. In particular for (4.28), we choose $\psi(u_1,u_2,u_3) = \frac{\lambda u_1^{(n+1)\alpha}}{\beta\Gamma((n+1)\alpha)((n+1)\alpha+1)}$ for $u_1 \ge 0$, $u_2 \ge 0$, $u_3 \ge 0$ and $\lambda, \alpha$ are given in Section 4. Similar choices for the other examples of Section 4. It is also worth noticing that estimate (4.2) derived in Section 4 is of independent interest but not needed in Theorem 2.1.

## 3 Semi-local convergence for explicit methods

A specialization of Theorem 2.1 can be utilized to study the semi-local convergence of the explicit methods given in the introduction of this study. In particular, for the study of method (1.12) (and consequently of method (1.11)), we use the conditions $(S')$ :

$(s_1')$ $F : \Omega \subset B_1 \to B_2$ is continuous and $A(x,x) \in \mathcal{L}(B_1, B_2)$ for each $x \in \Omega$.

$(s_2')$ There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,x)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $x \in \Omega_0$ and

$$\left\| A(x,x)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

$(s_3')$ There exist continuous and nondecreasing functions $\psi_0 : [0,+\infty)^3 \to [0,+\infty)$, $\psi_2 : [0,+\infty)^3 \to [0,+\infty)$ with $\psi_0(0,0,0) = \psi_2(0,0,0) = 0$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(y,y)(x-y)\| \leq$$

$$\beta\psi_0(\|x-y\|, \|x-x_0\|, \|y-x_0\|)\|x-y\|$$

and

$$\|A(x,y) - A(y,y)\| \leq \beta\psi_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|).$$

Set $\psi = \psi_0 + \psi_2$.

$(s_4')$ There exist $x_0 \in \Omega_0$ and $\eta \geq 0$ such that $A(x_0,x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\left\| A(x_0,x_0)^{-1} F(x_0) \right\| \leq \eta.$$

$(s_5') = (s_6)$

$(s_6') = (s_7)$.

Next, we present the following semi-local convergence analysis of method (1.12) using the $(S')$ conditions and the preceding notation.

**Proposition 3.1.** *Suppose that the conditions $(S')$ are satisfied. Then, sequence $\{x_n\}$ generated by method (1.12) starting at $x_0 \in \Omega$ is well defined in $U(x_0,s)$, remains in $U(x_0,s)$ for each $n = 0,1,2,...$ and converges to a unique solution $x^* \in \overline{U}(x_0,s)$ of equation $F(x) = 0$.*

*Proof.* We follow the proof of Theorem 2.1 but use instead the analogous estimate

$$\|F(x_k)\| = \|F(x_k) - F(x_{k-1}) - A(x_{k-1},x_{k-1})(x_k - x_{k-1})\| \leq$$

$$\|F(x_k) - F(x_{k-1}) - A(x_k,x_{k-1})(x_k - x_{k-1})\| +$$

$$\begin{gathered}\left\|\left(A\left(x_k, x_{k-1}\right)-A\left(x_{k-1}, x_{k-1}\right)\right)\left(x_k-x_{k-1}\right)\right\| \leq \\ \left[\psi_0\left(\left\|x_k-x_{k-1}\right\|,\left\|x_{k-1}-x_0\right\|,\left\|x_k-x_0\right\|\right)+\right. \\ \left.\psi_2\left(\left\|x_k-x_{k-1}\right\|,\left\|x_{k-1}-x_0\right\|,\left\|x_k-x_0\right\|\right)\right]\left\|x_k-x_{k-1}\right\|= \\ \psi\left(\left\|x_k-x_{k-1}\right\|,\left\|x_{k-1}-x_0\right\|,\left\|x_k-x_0\right\|\right)\left\|x_k-x_{k-1}\right\| .\end{gathered}$$

The rest of the proof is identical to the one in Theorem 2.1 until the uniqueness part for which we have the corresponding estimate

$$\begin{gathered}\left\|x^{**}-x_{k+1}\right\|=\left\|x^{**}-x_k+A_k^{-1} F\left(x_k\right)-A_k^{-1} F\left(x^{**}\right)\right\| \leq \\ \left\|A_k^{-1}\right\|\left\|F\left(x^{**}\right)-F\left(x_k\right)-A_k\left(x^{**}-x_k\right)\right\| \leq \\ \beta^{-1} \beta \psi_0\left(\left\|x^{**}-x_k\right\|,\left\|x_{k-1}-x_0\right\|,\left\|x_k-x_0\right\|\right) \leq \\ q\left\|x^{**}-x_k\right\| \leq q^{k+1}\left\|x^{**}-x_0\right\| .\end{gathered}$$

□

**Remark 3.1.** Comments similar to the ones given in Section 2 can follows but for method (1.13) and method (1.14) instead of method (1.5) and method (1.6), respectively.

# 4 Applications to $X$-valued Fractional Calculus

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions $f$ of real domain $[a, b]$. All integrals are of Bochner-type, see [13]. The derivatives of $f$ are defined similarly to numerical ones, see [16], pp. 83-86 and p. 93.

Let $f : [a, b] \rightarrow X$ such that $f^{(m)} \in L_{\infty}([a, b], X)$, the $X$-valued left Caputo fractional derivative of order $\alpha \notin \mathbb{N}$, $\alpha > 0$, $m = \lceil\alpha\rceil$ ($\lceil\cdot\rceil$ ceiling) is defined as follows (see [3]):

$$\left(D_a^\alpha f\right)(x)=\frac{1}{\Gamma(m-\alpha)} \int_a^x(x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{4.1}$$

where $\Gamma$ is the gamma function, $\forall\, x \in [a, b]$.

We observe that

$$\left\|\left(D_a^\alpha f\right)(x)\right\| \leq \frac{1}{\Gamma(m-\alpha)} \int_a^x(x-t)^{m-\alpha-1}\left\|f^{(m)}(t)\right\| dt$$

$$\leq \frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha)}\left(\int_a^x(x-t)^{m-\alpha-1} dt\right)=\frac{\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha)} \frac{(x-a)^{m-\alpha}}{(m-\alpha)}$$

$$= \frac{\left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (x-a)^{m-\alpha}. \tag{4.2}$$

We have proved that

$$\|(D_a^{\alpha} f)(x)\| \leq \frac{\left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (x-a)^{m-\alpha} \leq \frac{\left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{4.3}$$

Clearly then $(D_a^{\alpha} f)(a) = 0$.

Let $n \in \mathbb{N}$ we denote $D_a^{n\alpha} = D_a^{\alpha} D_a^{\alpha} ... D_a^{\alpha}$ ($n$-times).

Let us assume now that

$$f \in C^1([a,b], X), \ D_a^{k\alpha} f \in C^1([a,b], X), \ k = 1, ..., n;$$

$$D_a^{(n+1)\alpha} f \in C([a,b], X), \ n \in \mathbb{N}, \ 0 < \alpha \leq 1. \tag{4.4}$$

By [4], we have

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} f\right)(a) + \tag{4.5}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_a^{(n+1)\alpha} f\right)(t)\, dt, \quad \forall\, x \in [a,b].$$

Under our assumption and conclusion, see (4.4), Taylor's formula (4.5) becomes

$$f(x) - f(a) = \sum_{i=2}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_a^{(n+1)\alpha} f\right)(t)\, dt, \quad \forall\, x \in [a,b], \text{ for } 0 < \alpha < 1. \tag{4.6}$$

Here we are going to operate more generally. Again we assume $0 < \alpha \leq 1$, and $f : [a,b] \to X$, such that $f' \in C([a,b], X)$. We define the following $X$-valued left Caputo fractional derivatives:

$$\left(D_y^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x (x-t)^{-\alpha} f'(t)\, dt, \tag{4.7}$$

for any $x \geq y$; $x, y \in [a,b]$, and

$$\left(D_x^{\alpha} f\right)(y) = \frac{1}{\Gamma(1-\alpha)} \int_x^y (y-t)^{-\alpha} f'(t)\, dt, \tag{4.8}$$

for any $y \geq x$; $x, y \in [a,b]$.

Notice $D_y^1 f = f'$, $D_x^1 f = f'$ by convention.

Clearly here $\left(D_y^{\alpha}f\right)$, $\left(D_x^{\alpha}f\right)$ are continuous functions over $[a,b]$, see [3]. We also make the convention that $\left(D_y^{\alpha}f\right)(x)=0$, for $x<y$, and $\left(D_x^{\alpha}f\right)(y)=0$, for $y<x$.

Here we assume that $D_y^{k\alpha}f$, $D_x^{k\alpha}f\in C^1\left([a,b],X\right)$, $k=1,...,n$; $D_y^{(n+1)\alpha}f$, $D_x^{(n+1)\alpha}f\in C\left([a,b],X\right)$, $n\in\mathbb{N}$; $\forall\ x,y\in[a,b]$.

Then by (4.6) we obtain

$$f(x)-f(y)=\sum_{i=2}^{n}\frac{(x-y)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_y^{i\alpha}f\right)(y)+$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_y^x(x-t)^{(n+1)\alpha-1}\left(D_y^{(n+1)\alpha}f\right)(t)\,dt, \tag{4.9}$$

$\forall\ x>y$; $x,y\in[a,b]$, for $0<\alpha<1$,

and also it holds

$$f(y)-f(x)=\sum_{i=2}^{n}\frac{(y-x)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_x^{i\alpha}f\right)(x)+$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^y(y-t)^{(n+1)\alpha-1}\left(D_x^{(n+1)\alpha}f\right)(t)\,dt, \tag{4.10}$$

$\forall\ y>x$; $x,y\in[a,b]$, for $0<\alpha<1$.

We define the following $X$-valued linear operator

$$(A(f))(x,y)=$$

$$\begin{cases}\sum_{i=2}^{n}\frac{(x-y)^{i\alpha-1}}{\Gamma(i\alpha+1)}\left(D_y^{i\alpha}f\right)(y)+\left(D_y^{(n+1)\alpha}f(x)\right)\frac{(x-y)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & x>y,\\ \sum_{i=2}^{n}\frac{(y-x)^{i\alpha-1}}{\Gamma(i\alpha+1)}\left(D_x^{i\alpha}f\right)(x)+\left(D_x^{(n+1)\alpha}f(y)\right)\frac{(y-x)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & y>x,\\ f'(x), \text{ when } x=y,\end{cases} \tag{4.11}$$

$\forall\ x,y\in[a,b], 0<\alpha<1$.

We may assume that

$$\|(A(f))(x,x)-(A(f))(y,y)\|=\|f'(x)-f'(y)\| \tag{4.12}$$

$$\leq\Phi|x-y|,\ \ \forall\ x,y\in[a,b],\text{ with }\Phi>0,$$

see also ([11], p. 3).

We estimate and have:

i) case of $x > y$ :

$$\|f(x) - f(y) - (A(f))(x,y)(x-y)\| =$$

$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} \left(D_y^{(n+1)\alpha} f\right)(t)\,dt \right. \tag{4.13}$$

$$\left. - \left(D_y^{(n+1)\alpha} f(x)\right) \frac{(x-y)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\|$$

(by [1], p. 426. Theorem 11.43)

$$= \frac{1}{\Gamma((n+1)\alpha)} \left\| \int_y^x (x-t)^{(n+1)\alpha-1} \left( \left(D_y^{(n+1)\alpha} f\right)(t) - \left(D_y^{(n+1)\alpha} f\right)(x) \right) dt \right\|$$

(by [7])

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} \left\| D_y^{(n+1)\alpha} f(t) - \left(D_y^{(n+1)\alpha} f\right)(x) \right\| dt$$

(we assume here that

$$\left\| D_y^{(n+1)\alpha} f(t) - D_y^{(n+1)\alpha} f(x) \right\| \leq \lambda_1 |t-x|, \tag{4.14}$$

$\forall\ t,x,y \in [a,b] : x \geq t \geq y$, where $\lambda_1 > 0$)

$$\leq \frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} (x-t)\,dt =$$

$$\frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha}\,dt = \frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(x-y)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{4.15}$$

We have proved that

$$\|f(x) - f(y) - (A(f))(x,y)(x-y)\| \leq \frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(x-y)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{4.16}$$

for any $x, y \in [a,b] : x > y, 0 < \alpha < 1$.

ii) case of $x < y$ :

$$\|f(x) - f(y) - (A(f))(x,y)(x-y)\| =$$

$$\|f(y) - f(x) - (A(f))(x,y)(y-x)\| =$$

$$\left\| \frac{1}{\Gamma((n+1)\alpha)} \int_x^y (y-t)^{(n+1)\alpha-1} \left(D_x^{(n+1)\alpha} f\right)(t)\,dt \right. \tag{4.17}$$

$$-\left(D_x^{(n+1)\alpha} f(y)\right) \frac{(y-x)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\Bigg\| =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left\| \int_x^y (y-t)^{(n+1)\alpha-1} \left( \left(D_x^{(n+1)\alpha} f\right)(t) - \left(D_x^{(n+1)\alpha} f\right)(y) \right) dt \right\| \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^y (y-t)^{(n+1)\alpha-1} \left\| \left(D_x^{(n+1)\alpha} f\right)(t) - \left(D_x^{(n+1)\alpha} f\right)(y) \right\| dt$$

(we assume that

$$\left\| \left(D_x^{(n+1)\alpha} f\right)(t) - \left(D_x^{(n+1)\alpha} f\right)(y) \right\| \leq \lambda_2 |t-y|, \tag{4.18}$$

$\forall\, t, y, x \in [a,b] : y \geq t \geq x$, where $\lambda_2 > 0$)

$$\leq \frac{\lambda_2}{\Gamma((n+1)\alpha)} \int_x^y (y-t)^{(n+1)\alpha-1} (y-t)\, dt =$$

$$\frac{\lambda_2}{\Gamma((n+1)\alpha)} \int_x^y (y-t)^{(n+1)\alpha}\, dt = \frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(y-x)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{4.19}$$

We have proved that

$$\|f(x) - f(y) - A(f)(x,y)(x-y)\| \leq \frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(y-x)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{4.20}$$

$\forall\, x, y \in [a,b] : y > x, 0 < \alpha < 1.$

**Conclusion** Let $\lambda := \max(\lambda_1, \lambda_2)$. It holds

$$\|f(x) - f(y) - (A(f))(x,y)(x-y)\| \leq \frac{\lambda}{\Gamma((n+1)\alpha)} \frac{|x-y|^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{4.21}$$

$\forall\, x, y \in [a,b]$, where $0 < \alpha < 1$, $n \in \mathbb{N}$.

One may assume that $\frac{\lambda}{\Gamma((n+1)\alpha)} < 1$.

(Above notice that (4.21) is trivial when $x = y$.)

Now based on (4.12) and (4.21), we can apply our numerical methods presented in this article, to solve $f(x) = 0$.

To have $(n+1)\alpha + 1 \geq 2$, we need to take $1 > \alpha \geq \frac{1}{n+1}$, where $n \in \mathbb{N}$.

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