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Classification of Some Special Types Ruled Surfaces in Simply Isotropic 3-Space

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Abstract. In this paper, we classify two types ruled surfaces in the three dimensional simply isotropic space $\mathbb{I}_3^1$ under the condition $\Delta\mathbf{x}_i = \lambda_i\mathbf{x}_i$ where Δ is the Laplace operator with respect to the first fundamental form and λ is a real number. We also give explicit forms of these surfaces.

AMS Subject Classification (2000).
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1 Introduction

Let $\mathbf{x} : \mathbf{M} \rightarrow \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively. Takahashi ([17]) proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta\mathbf{x} = \lambda\mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$.

As an extension of Takahashi theorem, in [8] Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition $\Delta\mathbf{x} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} \in Mat\ (m, \mathbb{R})$ is an $m \times m$-diagonal matrix, and proved that such hypersurfaces are minimal ($\mathbf{H} = 0$) in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen ([6]) investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition $\Delta\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}$,where $\mathbf{A} \in Mat\ (3, \mathbb{R})$ is a 3×3-real matrix and $\mathbf{B} \in \mathbb{R}^3$.

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen ([7]) studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition $\Delta\mathbf{G} = \mathbf{A}\mathbf{G}$,where $\mathbf{A} \in Mat$ $(3, \mathbb{R})$. Baikoussis and Verstraelen ([3]) studied the helicoidal surfaces in $\mathbb{E}^3$. Yoon ([19, 20]) classified the surfaces of revolution and the translation surfaces in the 3-dimensional Galilean space and pseudo-Galilean 3-space under the condition $\Delta\mathbf{x}^i = \lambda^i x^i$ and $\Delta\mathbf{r}_i = \lambda_i\mathbf{r}_i$,where $\lambda^i \in \mathbb{R}$. Karacan and Yoon ([10, 11]) classified translation surfaces and helicoidal surfaces in the three-dimensional simply isotropic space $\mathbb{I}_3^1$.

Kamenarović ([9]) studied the natural geometry of ruled surfaces and defined equations for the three types ruled surfaces in simply isotropic space $\mathbb{I}_3^1$. Sipus and Divjak ([15]) studied some mappings of skew ruled surfaces in simply isotropic space which preserve the generators.

The main purpose of this paper is to complete classification of special non-developable ruled surfaces of Type 3 and Type 4 defined by W.Vogel in the three dimensional simply isotropic space $\mathbb{I}_3^1$ in terms of the position vector field and the Laplacian operator.

2 Preliminaries

Motions and metric Isotropic geometry is based on the following group $\mathbf{G}_6$ of affine transformations $(x, y, z) \to (x', y', z')$ in $\mathbb{R}^3$,

$$\begin{aligned} x' &= a + x\cos\theta - y\sin\theta \\ y' &= b + x\sin\theta + y\cos\theta \\ z' &= c + +c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $a, b, c, c_1, c_2, \theta \in R$. Such affine transformations are called isotropic congruence transformations or isotropic motions . We see that isotropic motions appear as Euclidean motions (a translation and a rotation) in the projection onto the $xy-$plane the result of this projection, $P = (x, y, z) \to P' = (x, y, 0)$ is called the "top view" ([11]). Hence, an isotropic motion is composed of a Euclidean motion in the $xy-$plane and an affine shear transformation in the $z-$direction.

On the other hand, the isotropic distance of two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is defined as the Euclidean distance of the top views, i.e.,

$$d\,(P, Q)_i = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{2.2}$$

Let $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$ be vectors in $\mathbb{I}_3^1$. The isotropic inner product of X and Y is defined by

$$\langle X, Y \rangle_i = \begin{cases} z_1 z_2, & \text{if} \quad x_i = y_i = 0, \\ x_1 x_2 + y_1 y_2, & \text{if} \quad \text{otherwise.} \end{cases} \tag{2.3}$$

We call a vector of the form $X = (0, 0, z)$ in $\mathbb{I}_3^1$ an isotropic vector, and a non-isotropic vector otherwise. Consider a C^r-surface $\mathbf{M}$, $1 \leq r$, in $\mathbb{I}_3^1$ parameterized by

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))\,.$$

A surface $\mathbf{M}$ immersed in $\mathbb{I}_3^1$ is called admissible if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces ([21]).

For such a surface, the coefficients E, F, G of its first fundamental form are calculated with respect to the induced metric and the coefficients L, M, N of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The first and the second fundamental form of $\mathbf{M}$ are defined by

$$\begin{aligned} I &= Edu^2 + Fdudv + Gdv^2, \\ II &= Ldu^2 + Mdudv + Ndv^2, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_i \ , F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_i \ , G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_i \ , \\ L &= \frac{\det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}\right)}{\sqrt{EG - F^2}}, M = \frac{\det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv}\right)}{\sqrt{EG - F^2}}, N = \frac{\det\left(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv}\right)}{\sqrt{EG - F^2}}. \end{aligned} \tag{2.3}$$

Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$. The isotropic unit normal vector field is given by $\mathbf{U} = (0, 0, 1)\,.$

The isotropic curvature $\mathbf{K}$ and the isotropic mean curvature $\mathbf{H}$ are defined by

$$\mathbf{K} = \frac{LN - M^2}{EG - F^2}, \quad 2\mathbf{H} = \frac{EN - 2FM + GL}{EG - F^2}. \tag{2.6}$$

The surface $\mathbf{M}$ is said to be isotropic flat (resp. isotropic minimal), if $\mathbf{K}$ (resp. $\mathbf{H}$) vanishes ([1, 10, 12, 16, 21]).

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators Δ of the first fundamental form on $\mathbf{M}$ are defined by ([2,4])

$$\Delta\mathbf{x} = -\frac{1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial u} \left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{EG - F^2}} \right) - \frac{\partial}{\partial v} \left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{EG - F^2}} \right) \right]. \tag{2.7}$$

3 Ruled Surfaces in $\mathbb{I}_3^1$

Let $\mathbf{M}$ be ruled surface in $\mathbb{I}_3^1$ given by the parametrization

$$\begin{aligned} \mathbf{x} \quad &: \quad I \times \mathbb{R} \rightarrow \mathbb{I}_3^1 \\ (u, v) \quad &\rightarrow \quad \mathbf{x}(u, v) = \alpha(u) + v\beta(u). \end{aligned} \tag{3.1}$$

We call the base curve α and the director curve β,where α is a differentiable curve parametrized by its arc length, i.e., $\langle \alpha', \alpha' \rangle_i = 1$ and $\langle \beta, \beta \rangle_i = 1$. The curve β is orthogonal to the tangent vector field T_α of the base curve α, i.e., $\langle \beta', T_\alpha \rangle = 0$. First of all, we consider non isotropic plane curves α and β parametrized by $\alpha(u) = (u, 0, f(u))$ and $\beta(u) = (0, 1, g(u))$ .Then the surface $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = (u, v, f(u) + vg(u)). \tag{3.2}$$

We consider isotropic curve $\alpha = (0, 0, f(u))$ and non isotropic space curve β parametrized by $\beta(u) = (\cos u, \sin u, g(u))$,where $\langle \beta, \beta \rangle_i = 1$. Then the surface $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, f(u) + vg(u)). \tag{3.3}$$

The functions f and g are smooth functions of one variable. We call the surfaces given by (3.2) and (3.3) as ruled surfaces of Type 3 and Type 4 in the three dimensional simply isotropic space $\mathbb{I}_3^1$, respectively ([12, 18]).

4 Ruled Surfaces of Type 3 Satisfying $\Delta \mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify the ruled surface of Type 3 in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta \mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta \mathbf{x} = (\Delta \mathbf{x}_1, \Delta \mathbf{x}_2, \Delta \mathbf{x}_3),$$

where

$$\mathbf{x}_1 = u,\ \mathbf{x}_2 = v,\ \mathbf{x}_3 = f(u) + vg(u).$$

For the ruled surface given by (3.2), the coefficients of the first and second fundamental form are

$$E = 1, F = 0, G = 1, \tag{4.2}$$

$$L = 0,\ M = g',\ N = f'' + vg'', \tag{4.3}$$

respectively. The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = -g'^2,\quad \mathbf{H} = \frac{f'' + vg''}{2}, \tag{4.4}$$

respectively.

Proposition 4.1. *The Ruled surface given by (3.2) in the three dimensional simply isotropic space $\mathbb{I}_3^1$ are isotropic flat or developable* $(\mathbf{K} = 0)$, *iff* $g(u) = c_1$ *for constant* c_1.

Suppose that the surface has non zero the Gaussian curvature, so $g'(u) \neq 0$. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3.2) and (2.7) turns out to be

$$\Delta \mathbf{x}_i = (0, 0, -f''(u) - vg''(u)). \tag{4.5}$$

Suppose that $\mathbf{M}$ satisfies (4.1). Then from (4.5), we have

$$(f''(u) + vg''(u)) = -\lambda (f(u) + vg(u)), \tag{4.6}$$

where $\lambda \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 1-type. First of all, we assume that $\mathbf{M}$ satisfies the condition $\Delta \mathbf{x}_i = 0$. We call a surface satisfying that condition a harmonic surface or isotropic minimal. In this case, we get from (4.6)

$$f''(u) + vg''(u) = 0. \tag{4.7}$$

The general solutions of the equation (4.7) with respect to f and g are given by

$$\begin{aligned} f(u) &= c_1u + c_2 - vg(u), \\ g(u) &= c_3u + c_4 - \frac{f(u)}{v}, \end{aligned}$$

where $c_i \in \mathbb{R}$. Here, the functions f and g are related. Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(u)$. For example, if we choose $f(u) = \ln u$, we have $g(u) = c_3u + c_4 - \frac{\ln u}{v}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = \left(u, v, \ln u + v\left(c_3u + c_4 - \frac{\ln u}{v}\right)\right). \tag{4.8}$$

Theorem 4.2. *Let* $\mathbf{M}$ *be a ruled surface given by (3.2) in* $\mathbb{I}_3^1$. *If* $\mathbf{M}$ *is harmonic or isotropic minimal, then it is congruent to an open part of the surface*

$$\mathbf{x}(u, v) = \left(u, v, f(u) + v\left(c_3u + c_4 - \frac{f(u)}{v}\right)\right).$$

If $\lambda \neq 0$, from (4.6), we have

$$(f''(u) + \lambda f(u)) + v\left(g''(u) + \lambda g(u)\right) = 0). \tag{4.9}$$

This equations are second order linear differential equations with constant coefficients.We discuss two cases according to constant λ.

Case 1: Let $\lambda > 0$, from (4.9), we obtain

$$\begin{aligned} f(u) &= c_1 \cos u\sqrt{\lambda} + c_2 \sin u\sqrt{\lambda} - vg(u), \\ g(u) &= c_3 \cos u\sqrt{\lambda} + c_4 \sin u\sqrt{\lambda} - \frac{f(u)}{v}, \end{aligned}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$. Here, the functions f and g are related. Based on the selection of the function $f(u)$ or $g(u)$, it is possible to obtain other form of the function $g(u)$ or $f(u)$. For example, if we choose $f(u) = \ln u$, In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = \left(u, v, \ln u + v\left(c_3 \cos u\sqrt{\lambda} + c_4 \sin u\sqrt{\lambda} - \frac{\ln u}{v}\right)\right). \tag{4.10}$$

Case 2: Let $\lambda < 0$, from (4.9), we obtain

$$\begin{aligned} f(u) &= c_1e^{u\sqrt{\lambda}} + c_2e^{-u\sqrt{\lambda}} - vg(u), \\ g(u) &= c_3e^{u\sqrt{\lambda}} + c_4e^{-u\sqrt{\lambda}} - \frac{f(u)}{v}, \end{aligned}$$

where $c_i \in \mathbb{R}$. If we choose $f(u) = \ln u$, we have

$$\mathbf{x}(u,v) = \left(u, v, \ln u + v\left(c_3 e^{u\sqrt{\lambda}} + c_4 e^{-u\sqrt{\lambda}} - \frac{\ln u}{v}\right)\right). \tag{4.11}$$

Theorem 4.3. *Let* $\mathbf{M}$ *be a non harmonic ruled surface given by (3.2) in the three dimensional simply isotropic space* $\mathbb{I}_3^1$*. If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta\mathbf{x}_i = \lambda_i \mathbf{x}_i$*, where* $\lambda_i \in \mathbb{R}$*,* $i=1,2,3$*, then it is congruent to an open part of the following surfaces*

$$\mathbf{x}(u,v) = \left(u, v, f(u) + v\left(c_3 \cos u\sqrt{\lambda} + c_4 \sin u\sqrt{\lambda} - \frac{f(u)}{v}\right)\right)$$

and

$$\mathbf{x}(u,v) = \left(u, v, f(u) + v\left(c_3 e^{u\sqrt{\lambda}} + c_4 e^{-u\sqrt{\lambda}} - \frac{f(u)}{v}\right)\right).$$

5 Ruled Surfaces of Type 4 Satisfying $\Delta\mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify the ruled surface of Type 4 in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{5.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta\mathbf{x} = (\Delta\mathbf{x}_1, \Delta\mathbf{x}_2, \Delta\mathbf{x}_3),$$

where

$$\mathbf{x}_1 = v\cos u,\ \mathbf{x}_2 = v\sin u,\ \mathbf{x}_3 = f(u) + vg(u).$$

For the ruled surface given by (3.3), the coefficients of the first and second fundamental form are

$$E = v^2, F = 0, G = 1, \tag{5.2}$$

$$L = -f'' - vg'' - vg,\ M = g',\ N = 0, \tag{5.3}$$

respectively. The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = -\frac{g'^2}{v^2},\quad \mathbf{H} = -\frac{f'' + v(g'' + g)}{2v^2}, \tag{5.4}$$

where $v \neq 0$,respectively.

Proposition 5.1. *The Ruled surface given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$ are isotropic flat or developable* ($\mathbf{K}=0$), *iff* $g(u)=c_1$ *for constant* c_1.

Suppose that the surface has non zero the Gaussian curvature, so $g'(u)\neq 0$. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3.3) and (2.7) turns out to be

$$\Delta\mathbf{x}_i=\left(0,0,-\frac{f''+v\left(g''+g\right)}{v^2}\right). \tag{5.5}$$

Suppose that $\mathbf{M}$ satisfies (5.1). Then from (5.5), we have

$$\frac{f''+v\left(g''+g\right)}{v^2}=-\lambda\left(f+vg\right), \tag{5.6}$$

where $\lambda\in\mathbb{R}$. This means that $\mathbf{M}$ is at most of 1-type. First of all, we assume that $\mathbf{M}$ satisfies the condition $\Delta\mathbf{x}_i=0$. We call a surface satisfying that condition a harmonic surface or isotropic minimal. In this case, we get from (5.6)

$$f''(u)+v\left(g''(u)+g(u)\right)=0. \tag{5.7}$$

The general solutions of the equation (5.7) with respect to f and g are given by

$$\begin{aligned} f(u) &= c_1u+c_2+\int_1^u\left(-v\int_1^z\left(g''(s)+g(s)\right)ds\right)dz,\\ g(u) &= c_3\cos u+c_4\sin u+\cos u\left(\int_1^u\frac{f''(x)\sin x}{v}dx\right)\\ &\quad -\sin u\left(\int_1^u\frac{f''(y)\sin y}{v}dy\right), \end{aligned}$$

where $c_i\in\mathbb{R}$. Here, the functions f and g are related. Based on the selection of the function $f(u)$ or $g(u)$, it is possible to obtain other form of the function $g(u)$ or $f(u)$. For example, if we choose $f(u)=e^u$, we have

$$g(u)=\frac{-e^u+e\cos(1-u)+2c_3v\cos u+2c_4v\sin u+e^u\sin 2u-e\sin(1+u)}{2v}.$$

In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\begin{pmatrix} v\cos u,\\ v\sin u,\\ e^u+v\left(\frac{-e^u+e\cos(1-u)+2c_3v\cos u+2c_4v\sin u+e^u\sin 2u-e\sin(1+u)}{2v}\right) \end{pmatrix}. \tag{5.8}$$

Theorem 5.2. *Let* $\mathbf{M}$ *be a ruled surface given by (3.3) in* $\mathbb{I}_3^1$*. If* $\mathbf{M}$ *is harmonic or isotropic minimal, then it is congruent to an open part of the surface*

$$\mathbf{x}(u,v)=\begin{pmatrix} v\cos u, \\ v\sin u, \\ f(u)+v\left(c_3\cos u+c_4\sin u+\cos u\left(\int_1^u \frac{f''(x)\sin x}{v}dx\right)\right. \\ \left.-\sin u\left(\int_1^u \frac{f''(y)\sin y}{v}dy\right)\right) \end{pmatrix}.$$

If $\lambda \neq 0$, from (5.6), we have

$$\left(\frac{f''}{v^2}+\lambda f\right)+v\left(\frac{(g''+g)}{v^2}+\lambda g\right)=0. \tag{5.9}$$

This equations are second order linear differential equations with constant coefficients.We discuss two cases according to constant λ.

Case 1: $\lambda > 0$, (5.9) can be separated, we obtain

$$\left(\frac{f''}{v^2}+\lambda f\right)=0 \tag{5.10}$$

and

$$\left(\frac{(g''+g)}{v^2}+\lambda g\right)=0. \tag{5.11}$$

Therefore, we have

$$\begin{aligned} f(u) &= c_1\cos uv\sqrt{\lambda}+c_2\sin uv\sqrt{\lambda}, \\ g(u) &= c_3e^{u\sqrt{-(1+v^2\lambda)}}+c_4e^{-u\sqrt{-(1+v^2\lambda)}}, \end{aligned}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$ and $(1+v^2\lambda)<0$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\begin{pmatrix} v\cos u, \\ v\sin u, \\ \left(c_1\cos uv\sqrt{\lambda}+c_2\sin uv\sqrt{\lambda}\right)+v\left(c_3e^{u\sqrt{-(1+v^2\lambda)}}\right. \\ \left.+c_4e^{-u\sqrt{-(1+v^2\lambda)}}\right) \end{pmatrix} \tag{5.12}$$

Case 2: Let $\lambda < 0$, from (5.9), we obtain

$$\left(\frac{f''}{v^2}-\lambda f\right)=0 \tag{5.13}$$

and

$$\left(\frac{(g''+g)}{v^2}-\lambda g\right)=0. \tag{5.14}$$

Therefore, we have

$$\begin{aligned} f(u) &= c_1e^{uv\sqrt{\lambda}}+c_2e^{-uv\sqrt{\lambda}}, \\ g(u) &= c_3e^{u\sqrt{-1+v^2\lambda}}+c_4e^{-u\sqrt{-1+v^2\lambda}}, \end{aligned}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$ and $(-1+v^2\lambda)>0$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\begin{pmatrix} v\cos u, \\ v\sin u, \\ \left(c_1e^{uv\sqrt{\lambda}}+c_2e^{-uv\sqrt{\lambda}}\right)+v\left(c_3e^{u\sqrt{-1+v^2\lambda}}+c_4e^{-u\sqrt{-1+v^2\lambda}}\right) \end{pmatrix}. \tag{5.15}$$

Theorem 5.3. *Let* $\mathbf{M}$ *be a non harmonic ruled surface given by (3.3) in the three dimensional simply isotropic space* $\mathbb{I}_3^1$. *If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta\mathbf{x}_i=\lambda_i\mathbf{x}_i$, *where* $\lambda_i\in\mathbb{R}$, $i=1,2,3$, *then it is congruent to an open part of the surfaces (5.12) and (5.15).*

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