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On the Diophantine Equation $x^2 - kxy + ky^2 + ly = 0$, $l = 2^n$

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Abstract. We consider the Diophantine equation $x^2-kxy+ky^2+ly = 0$ for $l = 2^n$ and determine for which values of the odd integer k, it has a positive integer solution x and y.

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1 Introduction

There are many works about the Diophantine equations. Given k and l are integers. Marlewski and Zarzycki [4] considered equation

$$x^2 - kxy + y^2 + lx = 0 \tag{1}$$

for $l = 1$ and found that equation (1) has an infinite number of positive integer solutions if and only if $k = 3$. Keskin [3] investigated positive integer solutions of equation (1) for $l = -1, 1$. He proved that when $k > 3$ equation (1) with $l = 1$ has no positive integer solutions but equation (1) with $l = -1$ has positive integer solutions. Moreover, he showed that the equation $x^2 - kxy - y^2 \mp x = 0$ and $x^2 - kxy - y^2 \mp y = 0$ have positive solutions when $k \geq 1$. Yuan and Hu [5] considered equation (1) for $l \in \{1, 2, 4\}$. They determined the value of k in the case of equation (1) has an infinite number of positive integer solutions. They found that equation (1) for $l = 1$

has infinitely many integer solutions if and only if $k \neq 0$ and ± 1 which generalized the work of Marlewski and Zarzycki [4]. After that Hu and Le [1] considered equation (1) for non zero integer l. They characterized the value of positive integer k that makes equation (1) has infinitely many positive integer solutions. One year later Karaatli and Şiar [2] studied the Diophantine equation

$$x^2 - kxy + ky^2 + ly = 0 \tag{2}$$

for $l \in \{1, 2, 4, 8\}$.

In this paper, we will consider the Diophantine equation (2) for $l = 2^n$ where n is a non-negative integer.

2 New Results

In order to prove the main theorem, we need the following auxiliary lemmas.

Lemma 2.1. *Equation (2) has a solution if and only if the equation*

$$X^2 - kXY + kY^2 + LY = 0. \tag{3}$$

has a solution a certain integer $L|l$ and $gcd(Y, L) = 1$.

Proof. Let p be a prime such that $p|y$ in equation (2). Then $p|x^2$, whereupon $p|x$. Suppose that $p|gcd(y, l)$. Then $p|l$ and $p|y$. From above we have $p|x$. Let $x = px', y = py'$ and $l = pl'$ for some integers x', y' and l'. It follows that

$$p^2x'^2 - kp^2x'y' + kp^2y'^2 + l'y'p^2 = 0,$$

therefore

$$x'^2 - kx'y' + ky'^2 + l'y' = 0,$$

with $l'|l$. If $gcd(l', y') > 1$, we repeat the same process until $gcd(Y, L) = 1$ and

$$X^2 - kXY + kY^2 + LY = 0$$

and $L|l$. If equation (3) has a solution $(X, Y) \in \mathbb{N}^2$ with $L|l$, then let $a = \frac{l}{L}, x = aX$ and $y = aY$ for some integers X and Y. Then

$$x^2 - kxy + ky^2 + ly = 0$$

□

We can then without loss of generality suppose that $gcd(y, l) = 1$ in equation (2).

Lemma 2.2. *If (x, y) is a solution to (2) with $gcd(y, l) = 1$, then y is a square.*

Proof. Let (x, y) be a solution to (2), and p a prime such that $p^t || y$. Then $p^t | x^2$. If t is an odd number then $p^{\frac{t+1}{2}} | x$. Therefore $p^{\frac{3t+1}{2}} | xy$ and $p^{2t} | y^2$, whereupon $p^{t+1} | xy$ and $p^{t+1} | y^2$. Hence $p^{t+1} | ly$. But, $p \nmid l$, then $p^{t+1} | y$, which is a contradiction. Then, if $p^t || y$, t is even and y is a square. □

Karaatli and Şiar [2] studied equation (2), when $l \in \{1, 2, 4, 8\}$. In the next theorem, we will study equation (2) when $l = 2^n$ where n is a non-negative integer and k is an odd integer. We prove the following.

Theorem 2.3. *If $l = 2^n$ for non-negative integer n and k is an odd number, then equation (2) has a positive solution only if $k = 5$ and all solutions are $x = ab, y = a^2$ with $gcd(a, b) = 1$ and $(2b - 5a)^2 - 5a^2 = -4$.*

Proof. We solve equation (2), where $gcd(y, l) = 1$ and $l = 1$ or 2^n.

Case 1: If $l = 2^n, n \neq 0$. Lemma 2.2 implies that y is a square. Let $y = u^2$, then $u^2 | x^2$, i.e., $u | x$. Let $x = ut$, where u and $t \in \mathbb{N}$. Then, equation (2) implies that

$$u^2t^2 - kutu^2 + ku^4 + lu^2 = 0. \tag{4}$$

Therefore

$$t^2 - kut + ku^2 + l = 0.$$

Since $gcd(y, l) = 1$, then u is odd. Then $ku^2 + l$ is odd. The integer $t^2 - kut = t(t - ku)$ is even for every $t \in \mathbb{N}$, then $(t^2 - kut) + (ku^2 + l)$ is odd, and equation (4) has no solution.

Case 2: If $l = 1$. Theorem 3.1 in [2] implies $k = 5$ and the rest of the proof follows. □

Remark 2.4. From Theorem 2.3, we know that equation (2) has a solution when $l = 2^n$ for non-negative integer n and an odd integer k if and only if $k = 5$. The solutions (x, y) are obtained from the solutions of the generalized Pell equation

$$u^2 - 5v^2 = -4,$$

which is known to have infinitely many solutions (u, v).

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